On characterizations of graphs having large geodetic numbers

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Abstract. Let $G$ be a nontrivial connected graph. For two vertices $u$ and $v$ of a graph $G$, the interval of $u$ and $v$ denoted by $I(u, v)$ is the set containing all vertices lying on some $u-v$ geodesic in $G$. Here a $u-v$ geodesic is a path of length $d(u, v)$. If $S$ is a set of vertices of $G$, then $I(S)$ is the union of all sets $I(u, v)$ for vertices $u$ and $v$ in $S$.

Now, if $I(S) = V(G)$ then $S$ is called a geodetic set of $G$ and the geodetic number $g(G)$ is the minimum cardinality among the geodetic sets of a graph $G$.

In this research, we determine the geodetic number of complete multipartite graphs, wheels and cycles with one chord. Moreover, we characterize all connected graphs of order $n$ having geodetic number $n - 1$.

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1. Introduction

Let $G$ be a nontrivial connected graph with the vertex set $V(G)$, the distance $d(u, v)$ between two vertices $u$ and $v$ in $G$ is the minimum length of a $u-v$ path in $G$. A $u-v$ path of length $d(u, v)$ is called a $u-v$ geodesic and the interval $I(u, v)$ is defined to be the set of all vertices lying on some $u-v$ geodesic of $G$. Let $S$ be a nonempty subset of $V(G)$,

$$I(S) = \bigcup_{u,v\in S} I(u,v).$$

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If \( I(S) = V(G) \) then \( S \) is called a geodetic set of \( G \). A geodetic set of minimum cardinality is a minimum geodetic set and the cardinality of a minimum geodetic set is the geodetic number denoted by \( g(G) \).

The concept of the geodetic number of a graph was first introduced by Chartrand, Harary and Zhang [1] in 2001. We refer to the books Graphs & Digraphs [2] and A First Course in Graph Theory [3] for terminology and notation not defined here.

To illustrate the concept, we consider the graphs \( G_1 \) and \( G_2 \) of Figure 1.

The graph \( G_1 \) has geodetic number 2 as \( S_1 = \{w_1, y_1\} \) is the unique minimum geodetic set of \( G_1 \).

For the graph \( G_2 \), on the other hand, each 2- element subset \( S \) of the vertex set of \( G_2 \) has the property that \( I(S) \) is properly contained in \( V(G_2) \). Therefore \( g(G_2) \geq 3 \) and since \( S_2 = \{u_2, v_2, x_2\} \) is a geodetic set of \( G_2 \), \( g(G_2) = 3 \).

It is clear that if \( G \) is a nontrivial connected graph then \( 2 \leq g(G) \leq n \) and the only nontrivial connected graph of order \( n \) that obtain the largest possible geodetic number \( n \) is the complete graph \( K_n \).

Among the results presented in [1], the following will be useful for us.
Theorem 1.1 [1]. If $G$ is a nontrivial connected graph of order $n$ and diameter $d$, then
\[ g(G) \leq n - d + 1. \]

Theorem 1.2 [1]. For integers $r, s \geq 2$, $g(K_{r,s}) = \min\{r, s, 4\}$.

A vertex $v$ is an extreme vertex in a graph $G$ if the subgraph induced by its neighbors is complete.

The following proposition is obtained by observing that each extreme vertex $v$ is either the initial or terminal vertex of a geodetic set containing $v$.

Proposition 1.3 [1]. Every geodetic set of a graph contains its extreme vertices.

In the next section, we study the geodetic number of complete multipartite graphs. In order to do that we need to introduce an additional notation.

For two graphs $G_1$ and $G_2$, the join $G = G_1 + G_2$ of $G_1$ and $G_2$ has the vertex set $V(G) = V(G_1) \cup V(G_2)$ and the edge set
\[ E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1), v \in V(G_2)\}. \]

2. The geodetic number of complete multipartite graphs

We have seen in Theorem 1.2 that the geodetic number of a complete bipartite graph $K_{r,s}$ where $r, s \geq 2$ is at most 4.

In this section, we investigate a more general result, namely, the geodetic number of a complete multipartite graph. For $k \geq 2$ and positive integers $n_i$ where $i \in \{1, 2, \ldots, k\}$, $K_{n_1,n_2,\ldots,n_k}$ is a complete $k$-partite graph. In particular, if $k = 2$ then $K_{n_1,n_2}$ is a complete bipartite graph.
Now, if \(n_1 = n_2 = 1\) then \(G = K_2\) and so \(g(K_{n_1,n_2}) = 2\); while if \(n_1 = 1\) but \(n_2 \geq 2\) then \(g(K_{n_1,n_2}) = n_2\).

Moreover, if \(n_1, n_2 \geq 2\) then \(g(K_{n_1,n_2}) = \min\{n_1, n_2, 4\}\).

**Theorem 2.1.** For positive integers \(n_k\) where \(k \geq 2\) and \(1 \leq n_1 \leq n_2 \leq \ldots \leq n_k\),

\[
g(K_{n_1,n_2,...,n_k}) = \begin{cases} 
k & \text{if } 1 = n_k \\
n_k & \text{if } 1 = n_{k-1} < n_k \\
 \min\{n_i, 4\} & \text{if } 1 = n_{i-1} < n_i \text{ for some } i \text{, where } 2 \leq i \leq k - 1 \\
 \min\{n_1, 4\} & \text{if } 1 < n_1.
\end{cases}
\]

**Proof.** For each integer \(k \geq 2\), let \(G = K_{n_1,n_2,...,n_k}\), where \(1 \leq n_1 \leq n_2 \leq \ldots \leq n_k\).

Let \(V = V(G)\) and \(V_i\) the partite set of \(G\) with cardinality \(n_i\) for \(1 \leq i \leq k\).

We consider two cases according to the cardinality \(n_k\) of \(V_k\).

**Case 1.** \(n_k = 1\).

Since \(n_k = 1\), it follows that \(n_i = 1\) for all \(i\) where \(1 \leq i \leq k\) and so \(G = K_k\). Hence \(g(G) = k\).

**Case 2.** \(n_k \geq 2\).

**Subcase 2.1.** \(n_{k-1} = 1\).

Then \(G = K_{k-1} + \overline{K_{n_k}}\). Let \(S = V_k\). We see that \(I(S) = V\) and so \(g(G) \leq n_k\).

Now notice that for each \(v \in V_k\), \(v\) is an extreme vertex since \(G[N(v)] = K_{k-1}\). By Proposition 1.3, the vertex \(v\) is contained in every geodetic set of \(G\).

Thus \(g(G) \geq n_k\) and therefore \(g(G) = n_k\).
Subcase 2.2. \(1 = n_{i-1} < n_i\) for some \(i\) where \(2 \leq i \leq k-1\).

Observe in this case that \(k \geq 3\).

If \(n_i = 2\) then \(S = V_i\) is a geodetic set of \(G\) and since \(G\) is nontrivial, \(g(G) = 2 = n_i\).

If \(n_i = 3\) then again \(S = V_i\) is a geodetic set of \(G\) and so \(g(G) \leq 3\).

It remains to show if \(U\) is a 2-element subset of \(V\) then \(U\) is not a geodetic set of \(G\).

Let \(U = \{x, y\}\). If \(x\) and \(y\) are in different partite sets then \(I(U) = U \neq V\).

On the other hand if \(U \subset A\) for some \(i \leq j \leq k\) then \(I(U) = A'_j \cup U \neq V\), since \(n_j \geq 3\).

Hence there is no 2-element geodetic set. It follows that

\[g(G) = 3 = n_i.\]

Now, if \(n_i \geq 4\) then \(S = \{u_1, u_2, v_1, v_2\}\) where \(\{u_1, u_2\} \subset A_i\) and \(\{v_1, v_2\} \subset A_{i+1}\) is a geodetic set of \(G\). Thus \(g(G) \leq 4\).

Let \(U = \{x, y, z\}\) be a 3-element subset of \(V\).

We will show \(U\) is not a geodetic set of \(G\). If \(x, y\) and \(z\) are in different partite sets then \(I(U) = U \neq V\). If there are some \(s, t\) such that \(\{x, y\} \subset A_s\) and \(\{z\} \subset A_t\) then \(I(U) = A'_s \cup U \neq V\).

Finally, if \(\{x, y, z\} \subset A_j\) for some \(j \geq i\) then \(I(U) = A'_j \cup U \neq V\). Thus there is no 3-element geodetic set and so \(g(G) = 4 = \min\{n_i, 4\}\).

Subcase 2.3. \(n_1 > 1\).

We consider three cases; \(n_1 = 2\), \(n_1 = 3\) and \(n_1 \geq 4\). Replacing \(i\) by 1 and employing the arguments similar to those used in Subcase 2.2, we have that \(g(G) = \min\{n_i, 4\}\). 

\(\Box\)
If \( n_1 = 2 \) then \( S = V_1 \) is a geodetic set of \( G \) and since \( G \) is nontrivial, \( g(G) = 2 = n_1 \).

If \( n_1 = 3 \) then again \( S = V_1 \) is a geodetic set of \( G \) and so \( g(G) \leq 3 \).

It remains to show if \( U \) is a 2-element subset of \( V \) then \( U \) is not a geodetic set of \( G \).

Let \( U = \{x, y\} \). If \( x \) and \( y \) are in different partite sets then \( I(U) = U \neq V \).

On the other hand if \( U \subset A_j \) for some \( 1 \leq j \leq k \) then \( I(U) = A_j \cup U \neq V \), since \( n_j \geq 3 \). Hence there is no 2-element geodetic set.

It follows that \( g(G) = 3 = n_1 \).

Now, if \( n_1 \geq 4 \) then \( S = \{u_1, u_2, v_1, v_2\} \) where \( \{u_1, u_2\} \subset A_1 \) and \( \{v_1, v_2\} \subset A_2 \) is a geodetic set of \( G \). Thus \( g(G) \leq 4 \).

Let \( U = \{x, y, z\} \) be a 3- element subset of \( V \). We will show \( U \) is not a geodetic set of \( G \). If \( x, y \) and \( z \) are in different partite sets then \( I(U) = U \neq V \). If there are some \( s, t \) such that \( \{x, y\} \subset A_s \) and \( \{z\} \subset A_t \) then

\[
I(U) = A'_s \cup U \neq V.
\]

Finally, if \( \{x, y, z\} \subset A_j \) for some \( j \geq i \) then \( I(U) = A'_j \cup U \neq V \).

Thus there is no 3-element geodetic set and so \( g(G) = 4 = \min\{n_1, 4\} \).

3. The geodetic number of a wheel

A wheel of order \( n \), denoted by \( W_n \), is the join of the cycle \( C_{n-1} \) and the trivial graph \( K_1 \). If \( n = 4 \) then \( W_n = K_4 \) and so \( g(W_n) = 4 \).

For \( n \geq 5 \), the geodetic number of \( W_n \) is shown in the following theorem.

**Theorem 3.1.** For a positive integer \( n \geq 5 \), \( g(W_n) = \lceil \frac{n-1}{2} \rceil \).
Proof. Let \( n \geq 5 \) be a positive integer and \( W_n = C_{n-1} + K_1 \) where \( C_{n-1} = (v_1, v_2, \ldots, v_{n-1}, v_n = v_1) \) and \( V(K_1) = \{ w \} \).

We consider two cases depending on whether \( n \) is odd or \( n \) is even.

Case 1. \( n \) is even.

Let \( S = \{ v_i \mid 1 \leq i \leq n-1 \text{ and } i \text{ is odd} \} \). Thus, the cardinality of \( S \) is \( \frac{n}{2} \).

For each even integer \( i \) where \( 2 \leq i \leq n-2 \), the vertices \( w \) and \( v_i \) lies on a \( v_{i-1} - v_{i+1} \) geodesic.

Therefore, \( I(S) = V(W_n) \).

Case 2. \( n \) is odd.

Let \( S = \{ v_i \mid 1 \leq i \leq n-2 \text{ and } i \text{ is odd} \} \). Thus, the cardinality of \( S \) is \( \frac{n-1}{2} \).

For each even integer \( i \) where \( 2 \leq i \leq n-1 \), the vertices \( w \) and \( v_i \) lies on a \( v_{i-1} - v_{i+1} \) geodesic.

Therefore, \( I(S) = V(W_n) \).

From the two cases above, we obtain that \( g(W_n) \leq \left\lceil \frac{n-1}{2} \right\rceil \).

To show that 

\[ g(W_n) \geq \left\lfloor \frac{n-1}{2} \right\rfloor, \]

we let \( S \) be a subset of the vertex set of \( W_n \) such that 

\[ |S| \leq \left\lfloor \frac{n-1}{2} \right\rfloor - 1 = \left\lfloor \frac{n-3}{2} \right\rfloor. \]

Since 

\[ |V(C_{n-1})| = n - 1, \]

there exists an integer \( i \) where \( 1 \leq i \leq n - 1 \) such that \( v_i, v_{i+1} \notin S \).

Since \( \text{diam}(W_n) = 2 \), for any pair \( x, y \in S \), \( v_i, v_{i+1} \notin I(x, y) \) and so \( v_i, v_{i+1} \notin I(S) \). This implies that \( S \) is not a geodetic set of \( W_n \).

Hence the result follows. \( \square \)
4. The geodetic number of cycles with one chord

For a positive integer \( n \geq 4 \), let \( C_n = (v_1, v_2, v_3, \ldots, v_n, v_1) \) and \( d \) the diameter of \( C_n \). Thus \( d = \left\lfloor \frac{n}{2} \right\rfloor \) and another way to write \( C_n \) is

\[
C_n = (v_1, v_2, \ldots, v_d, v_{d+1}, \ldots, v_n, v_1).
\]

If \( e \notin E(C_n) \) then \( C_n + e \) is a cycle with a chord.

Without loss of generality, let \( e = v_1v_i \) where \( 3 \leq i \leq n - 1 \). If \( n \) is even then \( d = \frac{n}{2} \) and for each \( i \) where

\[
3 \leq i \leq d,
\]
\[
C_n + v_1v_i,
\]
\[
C_n + v_1v_{n+2-i}
\]

are isomorphic.

On the other hand, if \( n \) is odd then \( d = \frac{n-1}{2} \) and for each \( i \) where

\[
3 \leq i \leq d + 1
\]
\[
C_n + v_1v_i
\]

is isomorphic to \( C_n + v_1v_{n+2-i} \).

We show next that the geodetic number of \( C_n + e \) is 2 or 3 depending on the parity of \( n \).

**Theorem 4.1.** For a positive integer \( n \geq 4 \),

\[
g(C_n + e) = \begin{cases} 
2 & \text{if } n \text{ is even} \\
3 & \text{if } n \text{ is odd}.
\end{cases}
\]

**Proof.** Let \( n \geq 4 \) be a positive integer. Since \( C_n + e \) is not a complete graph, \( g(C_n + e) \geq 2 \).
First, we assume that $n$ is even. It suffices to show that for each $i$, where $3 \leq i \leq d+1$, $g(C_n + v_1v_i) = 2$.

Let $S = \{v_{\lfloor \frac{i}{2} \rfloor}, v_{d+\lfloor \frac{i}{2} \rfloor}\}$.

Observe that $d(v_{\lfloor \frac{i}{2} \rfloor}, v_{d+\lfloor \frac{i}{2} \rfloor}) = d$. Since $P$ and $P'$ where

$$P = (v_{\lfloor \frac{i}{2} \rfloor}, v_{\lfloor \frac{i}{2} \rfloor+1}, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{d+\lfloor \frac{i}{2} \rfloor-1}, v_{d+\lfloor \frac{i}{2} \rfloor}),$$

$$P' = (v_{\lfloor \frac{i}{2} \rfloor}, v_{\lfloor \frac{i}{2} \rfloor-1}, \ldots, v_2, v_1, v_n, \ldots, v_{d+\lfloor \frac{i}{2} \rfloor+1}, v_{d+\lfloor \frac{i}{2} \rfloor+1})$$

are both $v_{\lfloor \frac{i}{2} \rfloor} - v_{d+\lfloor \frac{i}{2} \rfloor}$ geodesics.

It follows that $I(v_{\lfloor \frac{i}{2} \rfloor}, v_{d+\lfloor \frac{i}{2} \rfloor}) = V(C_n + e)$.

Hence $I(S) = V(C_n + e)$ and so $g(C_n + e) = 2$ if $n$ is even.

Next, suppose that $n$ is odd.

Similarly, it is sufficient to show that for each $i$, where $3 \leq i \leq d+1$, $g(C_n + v_1v_i) = 3$.

Again, $d(v_{\lfloor \frac{i}{2} \rfloor}, v_{d+\lfloor \frac{i}{2} \rfloor}) = d$. If $U$ is a two-element subset of $V(C_n + e)$ then $I(U) \neq V(C_n + e)$. Thus $g(C_n + e) \geq 3$.

Let $S = \{v_{\lfloor \frac{i}{2} \rfloor}, v_{d+\lfloor \frac{i}{2} \rfloor}, v_{d+\lfloor \frac{i}{2} \rfloor+1}\}$.

Then $P$ and $P'$ where

$$P = (v_{\lfloor \frac{i}{2} \rfloor}, v_{\lfloor \frac{i}{2} \rfloor+1}, \ldots, v_{i-1}, v_i, v_{i+1}, \ldots, v_{d+\lfloor \frac{i}{2} \rfloor-1}, v_{d+\lfloor \frac{i}{2} \rfloor}),$$

$$P' = (v_{\lfloor \frac{i}{2} \rfloor}, v_{\lfloor \frac{i}{2} \rfloor-1}, \ldots, v_2, v_1, v_n, \ldots, v_{d+\lfloor \frac{i}{2} \rfloor+2}, v_{d+\lfloor \frac{i}{2} \rfloor+1})$$

are both $v_{\lfloor \frac{i}{2} \rfloor} - v_{d+\lfloor \frac{i}{2} \rfloor}$ geodesics.

It follows that $I(S) = V(C_n + e)$ and $S$ is a geodetic set of $V(C_n + e)$.

Thus $g(C_n + e) = 3$ if $n$ is odd.
5. Graphs with large geodetic number

Let $G$ be a graph obtained by joining $\bigcup_{i=1}^{k} \ell_{i}K_{i}$ and $K_{1}$ that is $G = \bigcup_{i=1}^{k} \ell_{i}K_{i} + K_{1}$ where $\ell_{i} \geq 0, k \geq 1$ and $\sum_{i=1}^{k} \ell_{i} \geq 2$. The graphs $G_{1} = 6K_{1} + K_{1}$ and $G_{2} = (K_{2} \cup 2K_{3} \cup K_{4}) + K_{1}$ are shown in Figure 2, for instance.

![Figure 2: Graphs $G_{1}$ and $G_{2}$](image)

Observe that the order of $G$ is $1 + \sum_{i=1}^{k} i\ell_{i}$ and $G$ contains exactly one cut vertex. If $u$ is the cut vertex of $G$, then $G - u$ contains at least 2 components since $\sum_{i=1}^{k} \ell_{i} \geq 2$ and each component is a complete graph $K_{i}$ for some $1 \leq i \leq k$.

The following lemma gives us an upper bound of the geodetic number of a graph containing $C_{4}$ or $C_{4} + e$ as an induced subgraph which will be useful for us to characterize all graphs of order $n \geq 3$ with geodetic number $n - 1$.

**Lemma 5.1.** If a connected graph $G$ of order $n$ contains $C_{4}$ or $C_{4} + e$ as an induced subgraph then $g(G) \leq n - 2$. 
Proof. First, we assume that $G$ contains $C_4 = (u, v, x, y, u)$ as an induced subgraph. Since $d(u, x) = 2$, $\{v, y\} \subseteq I(u, x)$.

Hence $S = V(G) - \{v, y\}$ is a geodetic set of $G$ and so $g(G) \leq n - 2$.

Next, we assume $G$ contains $C_4 + e$ as an induced subgraph.

Let $C_4 = (u, v, x, y, u)$ and $e = vy$. Since $d(u, x) = 2$, $\{v, y\} \subseteq I(u, x)$.

Hence $S = V(G) - \{v, y\}$ is a geodetic set of $G$ and so $g(G) \leq n - 2$. \qed

Theorem 5.2. If $G$ is a graph of order $n$ then $g(G) = n - 1$ if and only if $G = \bigcup_{i=1}^{k} \ell_i K_i + K_1$ where $\ell_i \geq 0, k \geq 1$ and $\sum_{i=1}^{k} \ell_i \geq 2$.

Proof. First, we assume $G = \bigcup_{i=1}^{k} \ell_i K_i + K_1$ where $\ell_i \geq 0, k \geq 1$ and $\sum_{i=1}^{k} \ell_i \geq 2$.

Let $u$ be the cut vertex of $G$. Then, as we mentioned, $G - u$ contains at least 2 components since $\sum_{i=1}^{k} \ell_i \geq 2$ and each component is a complete graph $K_i$ for some $1 \leq i \leq k$.

Thus, for each vertex $v \in G - u$, $v$ is an extreme vertex in $G$.

It follows from Theorem 1.3 that if $S$ is a geodetic set of $G$ then $v \in S$. Since the cardinality of the vertex set of $G - u$ is $n - 1$, $g(G) \geq n - 1$.

To show $g(G) \leq n - 1$, we observe that $\text{diam}(G) = 2$ and so by Theorem 1.1, the result follows.

For the converse, we let $G$ be a connected graph of order $n$ such that $g(G) = n - 1$. Then $G$ is not complete. Moreover, by Theorem 1.1, $1 \leq \text{diam}(G) \leq 2$.

Thus $\text{diam}(G) = 2$. 

Let $u$ and $v$ be two vertices of $G$ such that $d(u, v) = 2$ and let $P = (u, w, v)$ be a $u - v$ geodesic in $G$. There is no vertex in $V(G) - V(P)$ that is adjacent to exactly one of $u$ and $v$, for otherwise there is a path whose length greater than $P$ which is not possible.

Also, there is no vertex in $V(G) - V(P)$ that is adjacent to both $u$ and $v$ or adjacent to $u$, $w$ and $v$, for otherwise $G$ contains $C_4$ or $C_4 + e$ as an induced subgraph which is not possible by Lemma 5.1.

Let $U$, $V$ and $W$ be subsets of $V(G) - V(P)$ defined as follows:

\[
U = \{x \mid x \text{ is adjacent to } u \text{ and } w \text{ but not } v\}
\]

\[
W = \{x \mid x \text{ is adjacent to } w \text{ but not } u \text{ and } v\}
\]

\[
V = \{x \mid x \text{ is adjacent to } w \text{ and } v \text{ but not } u\}.
\]

It is clear that the sets $U, W$ and $V$ are pairwise disjoint. Moreover, the union of $U, W$ and $V$ is $V(G) - V(P)$ since $\text{diam}(G) = 2$.

We consider two cases according to the cardinality of $W$.

**Case 1.** $|W| = 0$.

**Subcase 1.1.** $|U| = |V| = 0$.

Then $G = P_3 = 2K_1 + K_1$.

**Subcase 1.2.** $|U| = 0$ and $|V| \geq 1$.

Then the subgraph of $G$ induced by $V \cup \{v\}$, $G[V \cup \{v\}]$, is connected.

We show that $G[V \cup \{v\}]$ is complete. If $|V| = 1$ then $G[V \cup \{v\}] = K_2$.

Thus we may assume $|V| \geq 2$.

If there exist $v_1$ and $v_2$ such that $v_1$ and $v_2$ are not adjacent in $G[V \cup \{v\}]$ then $G$ contains $C_4 + e$ as an induced subgraph which is not possible.

Thus $G[V \cup \{v\}] = K_i$ for some $i \geq 3$ and so $G = (K_1 \cup K_i) + K_1$. 

Subcase 1.3. $|U| \geq 1$ and $|V| = 0$.

The argument similar to Subcase 1.2 can be used to show that the subgraph of $G$ induced by $U \cup \{u\}$, $G[U \cup \{u\}]$, is connected and complete and so $G = (K_1 \cup K_j) + K_1$ for some positive integer $j$.

Subcase 1.4. $|U| \geq 1$ and $|V| \geq 1$.

The argument similar to Subcases 1.2 and 1.3 can be used to show that $G[U \cup \{u\}]$ and $G[V \cup \{v\}]$, are connected and complete and so $G = (K_i \cup K_j) + K_1$ for some positive integers $i$ and $j$.

Case 2. $|W| \geq 1$.

We claim that the subgraph of $G$ induced by $W$ is a complete graph or the union of complete graphs. If $|W| = 1$ then $G[W] = K_1$ and so the claim is true.

Now, we assume $|W| \geq 2$. If there is a component of $G[W]$ that is not complete then $G$ contains $C_4 + e$ as an induced subgraph which is not possible. Thus the claim follows.

As a consequence, if $|U| = |V| = 0$ then

$$G = \left( \bigcup_{i=1}^{k} \ell_i K_i \cup 2K_1 \right) + K_1,$$

and if $|U| = 0$ and $|V| \geq 1$ then

$$G = \left( \bigcup_{i=1}^{k} \ell_i K_i \cup K_1 \right) + K_1.$$

While if $|U| \geq 1$ and $|V| = 0$ then

$$G = \left( \bigcup_{i=1}^{k} \ell_i K_i \cup K_1 \right) + K_1.$$
and if $|U| \geq 1$ and $|V| \geq 1$ then

$$G = \bigcup_{i=1}^{k} \ell_i K_i + K_1.$$  

6. Conclusion

In this research, we have investigated the geodetic number of complete multipartite graphs, wheels and cycles with one chord. We determine that the geodetic number of complete multipartite graphs can be as large as we like while the geodetic number of wheels is at most half of their order and the geodetic number of cycles with one chord is either 2 or 3.

We also characterize all connected graphs of order $n$ having geodetic number $n - 1$.

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On characterizations of graphs

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