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A remark on commutative subalgebras of Grassmann algebra

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Abstract. In this paper we prove that an interesting combinatorial inequality holds true. The importance of this inequality is due to its implication on settling a conjecture on structure of maximal commutative subalgebras of Grassmann algebra, posted by Domokos and Zubor in 2015.

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1. Introduction

The Grassmann algebra (exterior algebra) $G(n)$ over a field F of characteristic different from two is the following finite dimensional associative algebra of rank n:

$$G(n) = F[x_1, \ldots, x_n] / < x_i x_j + x_j x_i \mid 1 \leq i, j \leq n >_F .$$

The Grassmann algebra is widely used in ring theory, differential geometry and the theory of manifolds. For example, the readers are invited to look at the reference paper [3].

It is clear that $\dim_F G(n) = 2^n$ and the identity $[[x, y], z] = 0$ is satisfied for all $x, y, z \in G(n)$. $G(n)$ has a large center and it is natural to investigate the structure of commutative subalgebras in $G(n)$. It was recently studied by Domokos and Zubor [2].

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When n is even, the structure of maximal commutative subalgebras (with respect to inclusion) in $G(n)$ is quite well-understood. In particular, Domokos and Zubor showed that all such maximal commutative subalgebras in $G(n)$ of even rank n have dimension $3 \cdot 2^{n-2}$ (Corollary 2.4, Theorem 7.1 (i) in [2]) despite the fact that not all of them are isomorphic (Theorem 7.1 (ii) in [2]).

However, the structure of maximal commutative subalgebras in $G(n)$ of odd rank n is less clear. Some partial results on the structure of maximal commutative subalgebras were obtained for $n = 5$ and $n = 7$ (Proposition 7.5 in [2]). Other than these numerical results, this topic was not studied thoroughly. In particular, the following conjecture was raised by Domokos and Zubor (Conjecture 7.3 in [2]):

Conjecture 1. *If $n = 4k + 1$ and A is a maximal commutative subalgebra of $G(n)$, then $\dim_F(A) \geq 3 \cdot 2^{n-2}$.*

In 2019, Bovdi and the first author [1] showed that this conjecture is false for $17 \leq n < 1000$, $n = 4k + 1$ and $k \geq 4$ (see Corollary 5.2 [1]). In this paper, the main result is to show that this conjecture is false for all n such that $n = 4k + 1$ and $k \geq 4$.

To achieve this goal, we only need to prove Theorem 1 stated below in Section 2.

2. The main theorem

Firstly, we begin by defining a quantity Q_k before stating the main theorem at the end of this section.

Let k be any positive integer such that $k \geq 2$.

We look at the following quantities:

$$C_1 := 7 \cdot \binom{4k+2}{2k} + \binom{4k+2}{2k+3}, \tag{1}$$

$$C_2 := \binom{4k+2}{2k+5} + \binom{7}{1} \cdot \binom{4k+2}{2k+4} + \binom{7}{2} \cdot \binom{4k+2}{2k+3} + 7 \cdot \binom{4k+2}{2k+2} + 28 \cdot \binom{4k+2}{2k+1} + \binom{7}{5} \cdot \binom{4k+2}{2k}, \tag{2}$$

$$C_3 := \sum_i \binom{4k+9}{i} \text{ for } i \geq 2k+7 \text{ and } i \text{ is odd.}$$

Let Q_k be the following quantity:

$$Q_k := \frac{C_1 + C_2 + C_3}{2^{4k+7}}. \tag{3}$$

The main goal is to prove the following theorem:

Theorem 1. *Let k be any positive integer. Then*

$$Q_k < 1.$$

Remark 1. Theorem 1 is essentially the same as Conjecture 5.1 in the paper [1] written by Bovdi and the first author, which is the result required to show that Conjecture 1 is false for all n such that $n = 4k+1$ and $k \geq 4$. (For more details, please refer paper [1] written by Bovdi and the first author.)

3. A proof of Theorem 1

We define the variable A as follows:

$$A := \frac{128 \cdot 16^k \cdot \left(\sqrt{\pi} \cdot \Gamma(2k+6) - 6k \cdot \Gamma\left(2k + \frac{9}{2}\right) - 13 \cdot \Gamma\left(2k + \frac{9}{2}\right)\right)}{\sqrt{\pi} \cdot (2k+5)!} \tag{4}$$

where $\Gamma(z)$ is the Gamma function. By the computer program Maple, it is shown that

$$C_3 = A. \tag{5}$$

We note that the well-known Gamma function $\Gamma(z)$ has the following property:

$$\Gamma(z+1) = z\Gamma(z)$$

where z is any complex number in the complex plane. Let k be any positive integer. We expand $\Gamma(2k+4.5)$ as follows:

$$\begin{aligned}
\Gamma\Big(2k+\frac{9}{2}\Big) &= \Big(2k+\frac{7}{2}\Big)\Big(2k+\frac{5}{2}\Big)\cdots\Big(\frac{1}{2}\Big)\Gamma\Big(\frac{1}{2}\Big) \\
&= \Big(2k+\frac{7}{2}\Big)\Big(2k+\frac{5}{2}\Big)\cdots\Big(\frac{1}{2}\Big)\sqrt{\pi} \\
&= \frac{1}{2^{2k+4}}\cdot(4k+7)(4k+5)\cdots 5\cdot 3\cdot 1\cdot\sqrt{\pi} \\
&= \frac{(4k+7)!}{2^{2k+4}\cdot(4k+6)(4k+4)\cdots 6\cdot 4\cdot 2}\cdot\sqrt{\pi} \\
&= \frac{(4k+7)!}{2^{4k+7}\cdot(2k+3)!}\cdot\sqrt{\pi}. \qquad (6)
\end{aligned}$$

By (6),

$$\begin{aligned}
&128\cdot 16^k\cdot\Big(\sqrt{\pi}\cdot\Gamma(2k+6) - 6k\cdot\Gamma\Big(2k+\frac{9}{2}\Big) - 13\cdot\Gamma\Big(2k+\frac{9}{2}\Big)\Big) \\
&= 2^{4k+7}\cdot\Big(\sqrt{\pi}\cdot(2k+5)! - 6k\cdot\frac{(4k+7)!}{2^{4k+7}\cdot(2k+3)!}\cdot \\
&\cdot\sqrt{\pi} - 13\cdot\frac{(4k+7)!}{2^{4k+7}\cdot(2k+3)!}\cdot\sqrt{\pi}\Big) \qquad (7) \\
&= \sqrt{\pi}\cdot\Big(2^{4k+7}\cdot(2k+5)! - 6k\cdot\frac{(4k+7)!}{(2k+3)!} - 13\cdot\frac{(4k+7)!}{(2k+3)!}\Big). \qquad (8)
\end{aligned}$$

By (7), we simplify the expression of A in (4) as follows:

$$\begin{aligned}
A &= 2^{4k+7} - \frac{(6k+13)\cdot(4k+7)!}{(2k+5)!\cdot(2k+3)!} \\
&= 2^{4k+7} - \frac{6k+13}{2k+5}\cdot\binom{4k+7}{2k+3}. \qquad (9)
\end{aligned}$$

By (1), (2), (3), (4), (5), (8), we write the expression Q_k as follows:

$$Q_k = 1 + \frac{D}{2^{4k+7}} - \frac{E}{2^{4k+7}} \qquad (10)$$

where the variables D and E are defined by

$$D := C_1 + C_2, \tag{11}$$

$$E := \frac{6k+13}{2k+5} \cdot \binom{4k+7}{2k+3}.$$

Theorem 1 is equivalent to the following theorem by (9):

Theorem 2. *Let k be any positive integer. Then*

$$D < E.$$

We simplify (10) as follows:

$$D = 35 \cdot \binom{4k+2}{2k} + 22 \cdot \binom{4k+2}{2k+3} + \binom{4k+2}{2k+5} + 7 \cdot \binom{4k+2}{2k+4} + 28 \cdot \binom{4k+2}{2k+1}.$$

We do the following algebraic manipulations on D which will be needed later:

$$D \cdot \frac{(2k)!}{(4k+2)!} = \frac{35}{(2k+2)!} + \frac{22 \cdot 2k}{(2k+3)!} + \frac{(2k)(2k-1)(2k-2)}{(2k+5)!} + \frac{7(2k)(2k-1)}{(2k+4)!} + \frac{28}{(2k+1)! \cdot (2k+1)}.$$

$$\begin{aligned} D \cdot \frac{(2k)!}{(4k+2)!} \cdot (2k+5)! = {} & 35(2k+5)(2k+4)(2k+3) \\ & + 22(2k)(2k+4)(2k+3) \\ & + (2k)(2k-1)(2k-2) + 7(2k) \\ & (2k-1)(2k+5) \\ & + \frac{28(2k+5)(2k+4)(2k+3)(2k+2)}{(2k+1)}. \end{aligned} \tag{12}$$

Similarly, we have the following expression for E,

$$\begin{aligned} & E \cdot \frac{(2k)!}{(4k+2)!} \cdot (2k+5)! \\ & = \frac{(6k+13)(4k+7)(4k+6)(4k+5)(4k+4)(4k+3)}{(2k+3)(2k+2)(2k+1)}. \end{aligned} \tag{13}$$

We multiply $(2k+1)(2k+2)(2k+3)$ to (11) and (12). The R.H.S. of these two equations become the following two expressions respectively:

$$35(2k+5)(2k+4)(2k+3)(2k+1)(2k+2)(2k+3)+22(2k)(2k+4)(2k+3)$$
$$(2k+1)(2k+2)(2k+3)+(2k)(2k-1)(2k-2)(2k+1)(2k+2)(2k+3)$$
$$+7(2k)(2k-1)(2k+5)(2k+1)(2k+2)(2k+3)$$
$$+28(2k+5)(2k+4)(2k+3)(2k+2)(2k+3)(2k+2), \tag{14}$$

and

$$(6k+13)(4k+7)(4k+6)(4k+5)(4k+4)(4k+3). \tag{15}$$

Let D' and E' be the expressions in (13) and (14) respectively. As polynomials in k, the dominating terms of D' and E' are $93 \cdot 2^6 \cdot k^6$ and $96 \cdot 2^6 \cdot k^6$ respectively.

Hence, it is clear that

$$D' < E' \text{ as } k \to \infty.$$

More precisely, we expand D' and E' algebraically to get the following two expressions:

$$\begin{aligned} D' &= 5952k^6 + 48672k^5 + 164816k^4 + 174552k^3 \\ &\quad + 294712k^2 + 154056k + 32760 \\ E' &= 6144k^6 + 51712k^5 + 177280k^4 \\ &\quad + 316640k^3 + 310496k^2 + 158328k + 32760. \end{aligned}$$

It is now clear that

$$D' < E' \text{ as } k > 0.$$

Finally, we note that the inequality $D' < E'$ is equivalent to the inequality $D < E$.

Hence, Theorem 2 is proved. And as a result, Theorem 1 is completely proved. □

5. Conclusion

In the paper [1] written by the V. Bovdi and the first author, the Conjecture 1 was partially proved to be false. In this paper, we provide the supplementary computation to show that the Conjecture was completely false for $n \geq 17$.

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Appendix

In the computer Program *Maple*, we simply type the following expression

$$F := \sum_{j=1}^{k+2} \binom{4k+9}{2k+5+2j},$$

then, this expression will be automatically converted by the program *Maple* as follows:

$$\frac{16^k \left(128\sqrt{\pi}\Gamma(6+2k) - 768\Gamma(2k+\frac{9}{2})k - 1664\Gamma(2k+\frac{9}{2})\right)}{\sqrt{\pi}\Gamma(6+2k)}.$$

We note that this last expression is the same as the expression for A in (4). The expression F is the same as the definition of the term C_3 in Section 2. As a result, we get the equation (5).