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A Survey of Totally Positive Matrix Completion Problems

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Abstract. An $m \times n$ partial matrix is a matrix in which some of its entries are known while the other entries are free to be selected from a specified set such as the set of real numbers. By completion of a matrix, we mean to find the values of the entries that are not specified so that the resultant completed matrix satisfies certain desired properties.

A matrix is said to be totally positive (TP)(resp. Totally non-negative(TN)) if every square submatrix of it has positive determinant(resp. non-negative determinant). So, in matrix completion problems, we ask which partial matrices have a completion with certain desired properties, whereas the TP completion problem demands which partial matrices have TP Completion. Since sub matrices of a TP matrix are also TP, in order to have a TP completion, the partial matrix must be partially TP.

The objective of this paper is to provide a comprehensive analysis of the methods of TP matrix completion problems.

This paper provides an overview of completion techniques for TP matrices, discussing existing approaches and analyzing their graph-theoretic characterizations. It focuses on the TP completion of specific patterns of unspecified entries in a partial matrix. The paper also characterizes the number of unspecified entries for rectangular matrices of order up to three.

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1. Introduction

1.1. Totally positive completion

An $m \times n$ partial matrix is a matrix in which some of its entries are known while the other entries are free to be selected from a specified set such as the set of real numbers. By completion of a matrix, we mean to find the values of the entries that are not specified so that the resultant completed matrix satisfies certain desired properties.

A matrix is said to be totally positive (TP)(resp.Totally non-negative (TN)) if every square submatrix of it has positive determinant (resp. non-negative determinant) [6, 7]. In matrix completion problems, we aim to identify which partial matrices can be completed while meeting particular desired criteria. On the other hand, the TP completion problem specifically investigates which partial matrices can be completed in a totally positive manner.

In this section, we present the essential background and preliminary results of TP matrix completion problems. The concept of totally positive matrices has its roots in the pioneering research of Gantmacher and Krain [10]. Their foundational work on this topic was elegantly consolidated in a revised version of their monograph.

Additionally, I.Schoenberg's influence led Karlin to publish an influential treatise titled 'Total Positivity' [22]. While the treatise primarily focuses on totally positive kernels, it also touches upon the discrete version - totally positive matrices. Karlin in his book [22] has developed the concept of total positivity and applied it to problems involving moment spaces, boundary value problems of higher order differential operators and problems in the theory of inequalities and convexity.

The class of totally positive matrices arises in Statistics, Mathematical Biology, Combinatorics, Dynamics, Approximation Theory, Operator theory, Geometry etc. [10, 20, 22, 23]. The close relation between the theory of spline functions and total positivity can be found in [11, 1]. There is now a significant amount of literature addressing various matrix completion problems [9, 3, 4, 17, 12, 19, 18, 14, 16, 15, 21].

A *pattern* of a partial matrix is a specific arrangement of the known entries and a pattern $\mathcal{P}$ is TP completable if every TP partial matrix containing pattern $\mathcal{P}$ is TP completable. In [7] it has been shown that not all partial TP matrices allow TP completion and it has shown that all $m \times n$ patterns with one specified entry are completable if and only if $min\{m, n\} = 1, 2, 3$. For $min\{m, n\} \geqslant 4$, there are only 12 positions which ensure completability namely the upper left and lower right triangles of six entries each.

An $m \times n$ matrix is denoted as $A = (a_{ij})$ and $A[\alpha, \beta]$ is a submatrix of A with row index is given by α and column index β where $\alpha \in \{1, 2, \ldots, m\}$ and $\beta \in \{1, 2, \ldots, n\}$. For $\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_k\} \subset \{1, 2, ..., n\}, 0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqslant n$, the dispersion of α is $d(\alpha) = \alpha_k - \alpha_1 - k + 1$. It gives the measure of how much the index set have been spread out relative to $\{1, 2, \ldots, n\}$. If $d(\alpha) = 0$, we say α is a contiguous index set. If α and β are contiguous index sets with $|\alpha| = |\beta| = k$, then the sub matrix $A[\alpha, \beta]$ is said to be a contiguous sub matrix of A and its minor is called a contiguous minor. If either α or β is $\{1, 2, \ldots, k\}$, $A[\alpha, \beta]$ is called an initial sub matrix and its minor an initial minor.

An important result in this regard is given by Fekete's criterion [8], which states that a matrix is totally positive if and only if every square sub matrix with contiguous row and column index has positive determinant.

The following (Vander monde matrix) is a classic example for a TP matrix.

In linear algebra, the Vander monde matrix, named after Alexandre – Theophile Vander monde is a matrix with the terms of a geometric progression in each row.

This $m \times n$ matrix

$$\begin{bmatrix} 1 & \alpha_1 & \alpha_1^2 & \dots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \alpha_2^2 & \dots & \alpha_2^{n-1} \\ . & & & & \\ 1 & \alpha_m & \alpha_m^2 & \dots & \alpha_m^{n-1} \end{bmatrix}$$

is totally positive if $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_m$.

In [12] a simple way to check the total positivity has been identified. The following theorem is a simple characterization of a TP matrix.

Theorem 1.1.1 [12]. *If all initial minors of a $m \times n$ matrix A are positive then the matrix is TP.*

This theorem implies that it is sufficient to check all the initial minors to determine whether a matrix is TP.

In the next section, we deal with TP completion of a matrix with one unspecified entry.

1.2. TP Completion of a matrix with one unspecified entry

The following example shows that a partial TP matrix with single unknown entry is not TP completable.

Example 1.1. Consider the 4×4 matrix with one unspecified entry in $(1,4)^{th}$ position

$$\begin{bmatrix} 1 & 1.0 & 0.3 & x \\ 0.3 & 1 & 1 & 0.3 \\ 0.1 & 8 & 1 & 1 \\ 0.01 & 0.1 & 0.3 & 1 \end{bmatrix}$$

In order to make the determinant positive, x must be less than $\frac{-163}{50}$. But x must be positive for TP completion. So, there is no TP completion for the above matrix.

The following lemma forms the basis for all theorems throughout this section.

Lemma 1.2.1 [7]. *Let* $A = \begin{bmatrix} x & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}$, $n \geqslant 2$ *be a partial TP in which* x *is the only unspecified entry. Then if* x *is chosen so that* $detA \geqslant 0$*, then* $detA[\{1, 2, \dots, k\}] > 0, k = 1, 2, \dots, n-1$.

The following result Theorem 1.2.2, can be used as a check based on the position of the unspecified entry.

Theorem 1.2.2 [7]. *For a* $n \times n$ *partial TP matrix* A *with* $n \geqslant 3$ *and* x*, the only unspecified entry in* (s, t) *position such that* $s+t \leqslant 4$ *or* $s+t \geqslant 2n-2$*,* A *has a TP completion.*

The positions of the unspecified entries corresponding to the conditions $s+t \leqslant 4$ or $s+t \geqslant 2n$–2 can be termed as good positions. The above result for square matrices can be extended to arbitrary $m \times n$ matrices with the help of following Lemmas 1.2.3 and 1.2.4

Lemma 1.2.3 [7]. *If* A *is an* $m \times n$ *TP matrix, then there exists a positive vector* $x \in \mathbb{R}^m$ *such that the augmented matrix* $[A|x]$ *is an* $m \times (n+1)$ *TP matrix. Similarly there exists positive vectors* $u \in \mathbb{R}^m, v, w \in \mathbb{R}^n$ *such that*$[u|A]$, $\begin{bmatrix} v \\ A \end{bmatrix}$, $\begin{bmatrix} A \\ w \end{bmatrix}$ *are all TP.*

Example 1.2. Consider the vector $x = (x_1, x_2, x_3)^T$ in $\mathbb{R}^3$ and the augmented matrix $[A|x]$

$$\begin{bmatrix} 1 & 1 & 1 & x_1 \\ 1 & 2 & 4 & x_2 \\ 1 & 3 & 9 & x_3 \end{bmatrix}$$

where A is the TP Vandermonde matrix.

We chose x_i's sequentially so that $[A|x][\{1,2,...,i\}\{1,2,3,4\}], i = 1,2,3$ is TP. For $i = 1, [A|x][\{1\}\{1,2,3,4\}]$ is the first row. So, choose $x_1 > 0$ arbitrarily say $x_1 = 2$.

Now choose x_2 such that $[A|x][\{1,2\}\{1,2,3,4\}]$ is TP. So x_2 can be chosen such that

$$\begin{vmatrix} 1 & 2 \\ 4 & x_2 \end{vmatrix} > 0$$

$$\begin{aligned} x_2 - 8 &> 0 \\ x_2 &> 8 \end{aligned}$$

Take $x_2 = 9$.

Now x_3 can be chosen such that $[A|x][\{1,2,3\}\{1,2,3,4\}]$ is TP. The contiguous minor involving x_3 are minor of $[A|x][\{2,3\}\{3,4\}]$ and $[A|x][\{1,2,3\}$ $\{2,3,4\}]$ which yield the conditions.

$$\begin{aligned} 4x_3 - 81 &> 0 \\ x_3 &> \frac{81}{4} \end{aligned}$$

and

$$\begin{vmatrix} 1 & 1 & 2 \\ 2 & 4 & 9 \\ 3 & 9 & x_3 \end{vmatrix} = 2x_3 - 42 > 0$$

This implies

$$\begin{aligned} 2x_3 &> 42 \\ x_3 &> 21 \end{aligned}$$

From these we can take $x_3 > 21$.

If $x_3 = 22$,the matrix is TP.

So, the completed matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 9 \\ 1 & 3 & 9 & 22 \end{bmatrix}$$

Lemma 1.2.4 [3]. *If A is an $m \times n$ partial TP matrix, then there exist positive vectors $x, u \in \mathbb{R}^m, v, w \in \mathbb{R}^n$ such that $[A|x]$, $[u|A]$, $\begin{bmatrix} v \\ A \end{bmatrix}$, $\begin{bmatrix} A \\ w \end{bmatrix}$ are all partial TP.*

The above two Lemmas 1.2.3 and 1.2.4 give sufficient conditions for a $m \times n$ partial TP with $3 \leqslant m \leqslant n$ with x as the only unknown entry is in the $(s,t)^{th}$ position to have a completion.

The following theorem gives a simple characterization for the TP completion of a rectangular matrix with number of rows less than 3, based on the position of unspecified entry.

Theorem 1.2.5 [7]. *Let A be an $m \times n$ partial TP matrix with $3 \leqslant m \leqslant n$ and with the only unspecified entry lying in $(s,t)^{th}$ position. Then A has a TP completion only if $s + t \leqslant 4$ or $s + t \geqslant m + n - 2$.*

Proof. If $s + t \leqslant 4$, by Lemma 1.2.4 we can add positive vectors below A one by one till we get an $n \times n$ TP matrix A_1. If $s + t \geqslant m + n - 2$, add positive vectors above A so that the unspecified entry x lies in the (s,t) position where $s + t \geqslant m + n - 2 = 2n - 2$.

So, by Theorem 1.2.2, we get a TP completion say A_2. Since A is the submatrix of the TP matrix A_1 or A_2, we get A as TP. □

The following theorem characterizes all TP completable matrices among all $2 \times n$ matrices.

Theorem 1.2.6 [7]. *If A is an $2 \times n$ partial TP matrix with exactly one unspecified entry, then A has a TP completion.*

Proof. Suppose the unspecified entry x lies in the $(1,k)$ position.

Let $A = \begin{bmatrix} a_1 & a_2 \dots & a_{k-1} & x & a_{k+1} & \dots & a_n \\ b_1 & b_2 \dots & b_{k-1} & b_k & b_{k+1} & \dots & b_n \end{bmatrix}$.
Since the matrix is partial TP, we have $a_i b_j - b_i a_j > 0$ for $i < j$. This implies, $a_i b_j > b_i a_j$. That is $\dfrac{a_i}{b_i} > \dfrac{a_j}{b_j}$.

So, it is enough to choose x such that $\dfrac{a_{k-1}}{b_{k-1}} > \dfrac{x}{b_k} > \dfrac{a_{k+1}}{b_{k+1}}$. □

The following lemma shows that, all rectangular partial matrices of order 3×4 with one unspecified entry is TP completable.

Lemma 1.2.7 [7]. *If A is a 3×4 partial TP with exactly one unspecified entry, then A has a TP completion.*

Proof. For a 3×4 matrix all the positions are good positions since they satisfy $s + t \leqslant 4$ or $s + t \geqslant 3 + 4 - 2 = 5$. □

The following theorem is similar to the previous Theorem 1.2.6 for matrices with more than three rows.

Theorem 1.2.8 [7]. *Let A be an $m \times n$ partial TP matrix such that $4 \leqslant m \leqslant n$ and in which the only unspecified entry lies in the (s, t) position. Then A has a TP completion iff $s + t \leqslant 4$ or $s + t \geqslant m + n - 2$.*

Proof. The sufficient condition follow from Theorem 1.2.5. To prove the necessary part assume $s+t \geqslant 5$ and $s+t \leqslant m+n-3$. An $m \times n$ matrix with this condition has a 4×4 submatrix with the unspecified entry in any one of the position $(1, 4), (4, 2), (2, 3)$ or $(3, 2)$. Suppose that (s, t) corresponds to the position $(1, 4)$.

We have shown in Example 1.1 that there exists a 4×4 partial TP matrix with the unspecified entry in $(1, 4)$ position is not TP completable. So by the lemmas above we can augment this matrix with positive row/-column vectors to get an $m \times n$ partial TP matrix whose unknown entry lies in (s, t) position with $5 \leqslant s + t \leqslant m + n - 3$. Since the partial 4×4 submatrix of$A$ does not have a TP completion, A also does not have a TP completion. □

The next subsection 1.3 completely deals with TP completion of partial matrices with exactly two unspecified entries.

1.3. TP Completion of a matrix with two unspecified entries

It is evident from Theorem 1.2.6 that a $2 \times n$ partial TP matrix with two unspecified entries can be easily completed. Also when $min\{m, n\} = 3$, the completable patterns are characterized in [7]. Except for the following four patterns, all 3×3 TP matrices with two unspecified entries are completable.

$$\begin{bmatrix} x & ? & x \\ ? & x & x \\ x & x & x \end{bmatrix}, \begin{bmatrix} x & x & x \\ x & x & ? \\ x & ? & x \end{bmatrix}, \begin{bmatrix} x & ? & x \\ x & x & ? \\ x & x & x \end{bmatrix}, \begin{bmatrix} x & x & x \\ ? & x & x \\ x & ? & x \end{bmatrix}$$

For 3×4 and 4×4 partial TP matrices the positions of pairs of unspecified entries that are not completable are given below.

For 3×4 partial TP matrix

(1,2)	(2,1),(2,3),(3,1),(3,3)
(1,3)	(2,2),(2,4)
(1,4)	(2,3),(3,3)
(2,1)	(1,2),(3,2)
(2,2)	(1,3),(3,1),(3,3)

For 4×4 partial TP matrix

(1,1)	(2,3)
(1,2)	(2,1),(2,3),(2,4),(3,1),(3,3),(3,4)
(1,3)	(2,2),(2,4),(3,1),(3,2)
(1,4)	(2,2), (2,3),(3,3)
(2,2)	(3,3)
(2,3)	(3,2),(3,4)

For larger values of m and n, such classification of possible pairs remains an open problem. In the next section, we survey the recent advances in TP completable patterns.

2. TP Completable Patterns

We survey the developments in TP matrix completion problems which includes the characterization of completion of $m \times n$ matrix with $3 \leqslant m \leqslant n$, with exactly one unspecified entry at certain positions called good positions , $3 \times n$ TP completable patterns, non completable pair of 3×4 and 4×4 partial TP matrices , patterns which are TP(TN) completable and graph theoretic techniques used in TP(TN) completion problems.

2.1. Patterns

A pattern $\mathcal{P}$ is TP completable if all partial TP matrix of $\mathcal{P}$ has a TP completion. Such patterns are called TP completable patterns. Certain completable TP/TN patterns are identified and discussed in [12]. There are patterns $\mathcal{Q}$ for which the partial TP matrices which do not have TP completion besides being TP, additional polynomial conditions are required for completion. The characterization of combinatorically symmetric TN completable patterns is given in [16]. They are the monotonically labeled block clique graphs.

The next Theorem 2.1.1 is a good characterization of TN completion in terms of graph theory.

Theorem 2.1.1 [16]. *Every partial TN matrix with specified entries corresponding to a monotonically labeled block clique graph, there is a TN completion.*

In [18], the analogous result for TP matrices is also proved.

Theorem 2.1.2 [18]. *If A is a $n \times n$ combinatorially symmetric partial TP matrix, with specified entries corresponding to a monotonically labeled block clique graph, there is a TP completion A' of A. Moreover if the blocks of A are symmetric, the completion A' may be taken to be symmetric.*

We know that it is possible to increase the $(1,1)$ and (m,n) entry without bound in a TP matrix and still being TP. Using this fact, the pattern such that if $(i,j)^{th}$ entry is unspecified then either (s,t) for $s \leqslant i$, and $t \leqslant j$ or for $s \geqslant i$ and $t \geqslant j$ can be completed to a TP matrix.

Let A be the pattern given below

$$\begin{bmatrix} ? & ? & ? & * & * & * \\ ? & ? & ? & * & * & * \\ ? & ? & ? & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

We can complete A by selecting a fairly large value for the bottom rightmost unspecified entry as it is the $(1,1)$ entry of the submatrix below and right to it. Continue this process moving to the left and completing each row.Similarly, this can be applied for the pattern with unspecified entries in bottom rightmost of the matrix. Patterns with a full row or column of unspecified entries are identified to have a TP completion [20].

Theorem 2.1.3 [20]. *Let A be a TP matrix. Then a line can be inserted between any pair of adjacent lines in A so that the resulting matrix is TP.*

A *tridiagonal* pattern has specified entries only on the main diagonal and the diagonals just above and below the main diagonal. All $m \times n$ tridiagonal patterns are TP completable.

2.2. P_1,P_1' and P_2 Patterns

An $m \times n$ pattern is called P_1 if it has just one unspecified entry in $(1,n)$ position and its transpose is called P_1' .These are called one-sided patterns. The $m \times n$ pattern is called P_2 if it has two unspecified entries in $(1,n)$ and $(m,1)$ positions which is also called the double sided pattern.Since an $m \times n$ matrix is TP if and only if its initial minors are positive, P_1 has a TP completion if and only if the minors of contiguous upper right entries are all positive. These minors are denoted by $UR_i(x)$.

$$UR_i(x) = UR_i(0) + x(-1)^{i+1} detA[\{2,\ldots,i\};\{n-i+1,\ldots,n-1\}.$$

$UR_i(x) > 0$ iff

$$x > -UR_i(0)/detA[\{2,\ldots,i\};\{n-i+1,\ldots,n-1\} \text{ for } i \text{ odd and}$$
$$x < UR_i(0)/detA[\{2,\ldots,i\};\{n-i+1,\ldots,n-1\} \text{ if } i \text{ is even.}$$

The lower bound of the first inequality is denoted by $URL_i(A)$ and the upper bound of the second inequality by $URU_i(A)$.

Now let $URL(A) = \max_{i=1,3,5,\ldots} URL_i(A)$ and $URU(A) = \min_{2,4,6,\ldots} URU_i(A)$.

If $m \leqslant n$ then $URL(A)$ and $URU(A)$ depends on last columns of A and if $m \geqslant n$ they depend upon the first n rows of A.The following characterization has been done regarding the TP completion of P_1 patterns.

Theorem 2.2.1 [21]. *If A is an $m \times n$ partial TP matrix of pattern P_1 , then A has a TP completion if and only if $URL(A) < URU(A)$.*

For the pattern P_1' of a partial TP matrix A, define $LLL(A) = URL(A^T)$ and $LLU(A) = URU(A^T)$ the result is as follows

Theorem 2.2.2 [21]. *If A is an $m \times n$ partial TP matrix of pattern P_1', then A has a TP completion iff $LLL(A) < LLU(A)$.*

If P_2 is an $m \times n$ partial TP matrix with $m \neq n$ we can consider this as two separate P_1 and P_1' problems. If $m = n$ the determinant is the only contiguous minor consisting of both unspecified entries. Let $A(x, y)$ be the partial TP matrix of pattern P_2 with the unspecified entry x in $(1, n)$ position and y in the $(n, 1)$ position. Then $A_{n1} = A[\{1, 2, \ldots, n-1\};\{2, 3, \ldots, n\}]$ is a partial TP matrix of pattern P_1 and $A_{1n} = A[\{2, 3, \ldots, n\};\{1, 2, \ldots, n-1\}]$ is a partial TP matrix of pattern P_1' .

Hence by the above two theorems we have the following result for the completion of P_2.

Theorem 2.2.3. *Let A be a partial TP matrix of pattern P_2. Then A has a TP completion if and only if*

(i) $URL(A_{n1}) < URU(A_{n1})$

(ii) $LLL(A_{1n}) < LLU(A_{1n})$ *and*

(iii) (a) $detA(URL(A_{n1}), LLL(A_{1n})) > 0$ *if n is odd and*

(b) $detA(URU(A_{n1}), LLU)A_{1n})) > 0$ *if n is even*

If an upper right sub matrix or lower left sub matrix is a P_1 or P_1' pattern which is not completable, then the bigger matrix is also not completable.

Example 2.1. Let $A(x) = \begin{bmatrix} 1.1 & 095 & 0.56 & x \\ 0.95 & 1.012 & 0.99 & 1 \\ 0.45 & 0.5 & 0.7 & 1 \\ 0.2245 & 0.25 & 0.6 & 1.1 \end{bmatrix}$. It is a partial TP matrix which has no TP completion as $detA(x) = -0.179850x - 0.002804$ which is always negative for$x \geqslant 0$.

Example 2.2. Let $A(x, y) = \begin{bmatrix} 1 & 1 & 0.4 & x \\ 0.4 & 1 & 1 & 4 \\ 0.2 & 0.8 & 1 & 1 \\ y & 0.2 & 0.4 & 1 \end{bmatrix}$. Then $A(x, y)$ is a partial TP matrix which is not TP completable as $detA(x, y) = -0.0016 - 0.0008x - 0.328y - 0.2xy$ which is always negative for $x, y \geqslant 0$.

From the above two examples we can say that the patterns P_1,P_1' and P_2 are TP completable only for $m, n \leqslant 3$.

2.3. Echelon and Jagged Patterns

For $m, n \geqslant 4$, additional polynomial conditions are required for TP completability. In the 4×4 case if either the north west, south east position or a position adjacent to already unknown position is also unknown, then such a pattern is completable. This leads to the definition of echelon and jagged patterns [21].

A pattern is said to be *echelon* if whenever a position is unspecified either all positions (1) north and east of it are unspecified or (2) south and west of it are unspecified.

$$\begin{array}{cccccc} * & * & * & ? & ? & ? \\ * & * & * & * & ? & ? \\ ? & * & * & * & * & ? \\ ? & ? & * & * & * & * \end{array}$$

A pattern is said to be *jagged* if whenever a position is unspecified either all positions (3) north and west of it are unspecified or (4) south and east of it are unspecified.

$$\begin{array}{cccccc} ? & ? & ? & * & * & * \\ ? & * & * & * & * & * \\ * & * & * & * & ? & ? \\ * & * & * & ? & ? & ? \end{array}$$

A pattern in which at least one of (1) or (2) and at least one of (3) or (4) occur is called a jagged echelon pattern.

Theorem 2.3.1 [21]. *An echelon pattern is TN completable if and only if it contains no* 4×4, P_1, P_1' *or* P_2 *as a sub pattern.*

Lemma 2.3.2 [21]. *A jagged pattern with just one unspecified entry is TP-completable.*

In a jagged pattern with one unspecified entry, the unspecified entry must be in the $(1,1)$ or (m,n) position. If it is in $(1,1)$ position,then as discussed in the beginning of this section , the unspecified entry can be chosen big enough that all minors it completes are all positive. A pattern is said to be singly jagged if either positions (3) or positions (4) are unspecified.

Lemma 2.3.3 [21]. *Any singly jagged pattern is TP-completable.*

Proof. If the pattern is upper left jagged, begin with the most south east unspecified entry and complete it by choosing all sub matrices and applying Lemma 2.2.2 Continuing this process with the next most south east entry, the upper left jagged partial TP can be completed. □

Since the jagged pattern is a combination of two singly jagged patterns we have the following result.

Theorem 2.3.4 [21]. *Each jagged pattern is completable.*

3. Conclusion

The totally positive matrices have their own applications in many areas of pure and applied mathematics such as Theory of Approximation, Statistics and stochastic process, Theory of integral operators etc. Many important properties and problems of TP/TN matrices are explored and recorded in [12] such as bi diagonal factorization, recognition and completion problems.

In this paper several patterns that have TP completion have been identified. For a combinatorially symmetric square matrix, the partial TN completions have been characterized as the monotonically labeled block clique graphs [8]. But in the case of TP completion, the result obtained is there is a TP completion if the connected graph of a combinatorially symmetric partial totally positive matrix is monotonically labeled block clique. For a non-symmetric non-square partial TP matrices, graph based characterizations is an open problem.

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