Journal of Analysis and Applications
Vol. 20 (2022), No.1, pp.69-80
ISSN: 0972-5954

URL : www.sasip.net

Direct sum of star matrices

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Abstract. Let S_n be the symmetric group of order n. The permanent of an $n \times n$ matrix $A = (a_{ij})$ is defined as $\sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}$. Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. A matrix $B \in \Omega_n$ is said to be a star matrix if $per(\alpha B + (1-\alpha)A \leq \alpha perB + (1-\alpha)perA$, for all $A \in \Omega_n$ and all $\alpha \in [0, 1]$. Karuppanchetty and Maria Arulraj [3] proposed the following two conjectures:

(i) The direct sum of two star matrices is a star (also known as the star conjecture).

(ii) The only stars in Ω_n are the direct sum of 2×2 star matrices and identity matrices upto permutations of rows and columns.

In this paper, we derive some sufficient conditions for the direct sum of matrices in Ω_2 to satisfy the inequality of the star conjecture. We also provide some classes of matrices in Ω_n which satisfy the star condition.

AMS Subject Classification (2020): 15A15

Keywords: Doubly stochastic matrices, permanents, star matrices

1. Introduction

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices and S_n be the symmetric group of order n. If $A = (a_{ij})$ is an arbitrary $n \times n$ matrix, then the permanent of A is a scalar valued function given by

$$perA = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma_{(i)}}.$$

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The permanent function has been studied extensively of date, and it is known that if $A \in \Omega_n$ $0 \leq perA \leq 1$. The direct sum of the matrices $A_i, 1 \leq i \leq n$, is defined as follows:

$$\bigoplus_{i=1}^{n} A_i = \text{diag}(A_1, A_2, ..., A_n) = \begin{pmatrix} A_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & A_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \\ \mathbf{0} & \mathbf{0} & \dots & A_n \end{pmatrix},$$

where $\mathbf{0}$ is the zero matrix.

It is natural to inquire whether permanent is a convex function on Ω_n, that is, to see the validity of the inequality

$$per(\alpha B + (1-\alpha)A) \leq \alpha perB + (1-\alpha)perA,$$

for all $A, B \in \Omega_n$ and for all $\alpha \in [0,1]$. That this is not the case in general was shown by Perfect [5]. However, in this paper, for $\alpha = \frac{1}{2}$ and $B = I$ the author showed that

$$per(\frac{1}{2}I + \frac{1}{2}A) \leq \frac{1}{2} + \frac{1}{2}perA \text{ for all } A \in \Omega_n.$$

Brualdi and Newman [1] improved this result by showing that

$$per(\alpha I_n + (1-\alpha)A) \leq \alpha + (1-\alpha)perA,$$

for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$.

Moreover, they established that the inequality

$$per(\alpha B + (1-\alpha)A) \leq \alpha perB + (1-\alpha)perA$$

will hold for all $\alpha \in [0,1]$ and for all $A, B \in \Omega_n$ iff for all $A, B \in \Omega_n$

$$\sum_{i,j=1}^{n} b_{ij} perA_{ij} \leq perB + (n-1)perA, \tag{1}$$

where $B = (b_{ij})$ and A_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of A and equality holds iff $A = B$. As $\alpha \in [0,1]$ the inequality (1) is also written as

$$\sum_{i,j=1}^{n} a_{ij} perB_{ij} \leq perA + (n-1)perB, \tag{2}$$

where $A = (a_{ij})$ and B_{ij} is the $(n-1)\times(n-1)$ matrix obtained by deleting the i^{th} row and j^{th} column of B.

Wang [7] called a matrix B in Ω_n a star, if B satisfies

$$per(\alpha B + (1-\alpha)A) \leq \alpha perB + (1-\alpha)perA,$$

for all $A \in \Omega_n$ and for all $\alpha \in [0,1]$.

According to the result of Brualdi and Newman [1] the necessary and sufficient condition for a matrix $B \in \Omega_n$ to be a star matrix is that it should satisfy inequality (1). The inequality (2) is equivalent to (1) as $\alpha \in [0,1]$.

Therefore the inequality (2) is also a necessary and sufficient condition for a matrix $B \in \Omega_n$ to be a star matrix. We call the inequality (2) as a star inequality.

Wang [7] proved that (i) every 2×2 doubly stochastic matrix is a star and (ii) if $B \in \Omega_n$ is a star then $perB \geq \dfrac{1}{2^{n-1}}$.

Karuppanchetty and Maria Arulraj [3] have disproved Wang's conjecture [7] which states that for $n \geq 3$ permutation matrices are the only stars, by proving the following matrix B to be a star matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 1-x \\ 0 & 1-x & x \end{pmatrix} = 1 \bigoplus \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix}$$

where $0 \leq x \leq 1$.

They also established that the only stars in Ω_3 are PBQ where B is as defined above and P and Q are permutation matrices.

Let

$$M(a,b,c,d)=\begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{pmatrix}\in\Omega_3.$$

The matrix $B = 1\bigoplus M(a,b;c,d) \in \Omega_4$ where $0 < a,b < 1$ and $a+b \neq 1$ is not a star since the only star in Ω_3 is $M(a,1-a;1-a,a)$ up to permutation of rows and columns. They [3] (quoted by Cheon and Wanless [2]) proposed the following two conjectures:

(i) The direct sum of two star matrices is a star (also known as the star conjecture).

(ii) The only stars in Ω_n are the direct sum of 2×2 star matrices and identity matrices upto permutations of rows and columns.

Both conjectures are still open for $n \geq 4$. Maria Arulraj and K. Somasundaram [6] derived a necessary condition for a matrix $B \in \Omega_n$ to be a star matrix. For integers r and $n, (1 \leq r \leq n)$, let $Q_{r,n}$ denote the set of all sequences $(i_1, i_2, ...i_r)$ such that $1 \leq i_1 ... < i_r \leq n$.

The following notations are defined by Minc [4].

For fixed $\alpha, \beta \in Q_{r,n}$ let $A(\alpha,\beta)$ denote the submatrix of A obtained by deleting the rows α and the columns β of A, let $A[\alpha|\beta]$ denote the submatrix of A formed by the rows α and the columns β of A and let $T(A[\alpha,\beta])$ denote the sum of all the elements of the matrix $A[\alpha|\beta]$.

Let

$$S_r(A,B)=\sum_{\alpha,\beta\in Q_{r,n}} perA[\alpha/\beta]perB(\alpha/\beta).$$

The following Lemma is due to Maria Arulraj and Somasundaram [6] which gives the necessary condition for a matrix $B \in \Omega_n$ to be a star matrix.

Lemma 1.1. *Let $B \in \Omega_n$. If there exists an $n \times n$ matrix $E \neq 0$, such that* *the perturbation matrix* $B+E \in \Omega_n$ *and* $\sum_{k=0}^{n-2}(n-(k+1))S_k(B,E) < 0$, *then* B *is not a star.*

In this paper, we derive some sufficient conditions for the direct sum of matrices in Ω_2 to satisfy the inequality of the star conjecture. We also provide some classes of matrices in Ω_n which satisfy the star inequality (2).

2. Direct sum of star matrices

In this section, we prove the following theorems related to the first conjecture.

Theorem 2.1. *Let*

$$B = \bigoplus_{i=1}^{m} \begin{pmatrix} x_i & 1-x_i \\ 1-x_i & x_i \end{pmatrix}$$

where $\frac{1}{2} \leq x_i \leq 1 - \frac{1}{2m}$. *Then* B *satisfies (2) for all* $A \in \Omega_n$ *such that for all odd* p, $T(A[(p,p+1)/(p,p+1)] \leq 1$.

Proof.

$$\begin{aligned}
&\sum_{i,j=1}^{n} a_{ij} perB_{ij} - perA - (n-1)perB \\
&= a_{11}x_1(2x_2^2-2x_2+1)...(2x_m^2-2x_m+1) + a_{12}(1-x_1)(2x_2^2-2x_2+1) \\
&\cdots(2x_m^2-2x_m+1) + a_{21}(1-x_1)(2x_2^2-2x_2+1)...(2x_m^2-2x_m+1) \\
&+ a_{22}x_1(2x_2^2-2x_2+1)\cdots(2x_m^2-2x_m+1) + a_{33}x_2(2x_1^2-2x_1+1) \\
&\cdots(2x_m^2-2x_m+1) + a_{34}(1-x_2)(2x_1^2-2x_1+1)...(2x_m^2-2x_m+1) \\
&+ a_{43}(1-x_2)(2x_1^2-2x_1+1)\cdots(2x_m^2-2x_m+1) \\
&+ a_{44}x_2(2x_1^2-2x_1+1)\cdots(2x_m^2-2x_m+1) + \cdots - perA - (n-1) \\
&(2x_1^2-2x_1+1)...(2x_m^2-2x_m+1) \\
&\leq (a_{11}+a_{12}+a_{21}+a_{22})x_1(2x_2^2-2x_2+1)...(2x_m^2-2x_m+1) \\
&+ (a_{33}+a_{34}+a_{43}+a_{44})x_2(2x_2^2-2x_2+1)...(2x_m^2-2x_m+1)
\end{aligned}$$

$+\cdots - perA - (n-1)(2x_1^2-2x_1+1)...(2x_m^2-2x_m+1)$

$\leq x_1(2x_2^2-2x_2+1)\cdots(2x_m^2-2x_m+1) + x_2(2x_1^2-2x_1+1)$

$\cdots(2x_m^2-2x_m+1)\cdots + x_m(2x_1^2-2x_1+1)\cdots(2x_{m-1}^2-2x_{m-1}+1)$

$- perA - (n-1)(2x_1^2-2x_1+1)\cdots(2x_m^2-2x_m+1)$

$\leq (2x_2^2-2x_2+1)\cdots(2x_m^2-2x_m+1)[x_1 - \frac{n-1}{m}(2x_1^2-2x_1+1)]$

$+ (2x_1^2-2x_1+1)\cdots(2x_m^2-2x_m+1)[x_2 - \frac{n-1}{m}(2x_2^2-2x_2+1)]$

$+\cdots + (2x_1^2-2x_1+1)\cdots(2x_{m-1}^2-2x_{m-1}+1)$

$[x_m - \frac{n-1}{m}(2x_m^2-2x_m+1)] - perA.$

Now,

$$
\begin{aligned}
&x_i - \frac{n-1}{m}(2x_i^2-2x_i+1)\\
&= x_i - \frac{2m-1}{m}(2x_i^2-2x_i+1)\\
&= x_i + (-2+\frac{1}{m})(2x_i^2-2x_i+1) \leq x_i + (-2+\frac{1}{m})\frac{1}{2}\\
&= x_i - 1 + \frac{1}{2m} \leq 0
\end{aligned}
$$

since $\frac{1}{2} \leq x_i \leq 1 - \frac{1}{2m}$.

Therefore, the inequality (2) is satisfied. □

Theorem 2.2. *Let*

$$
B = \begin{pmatrix}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_1 & 1-x_1\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1-x_1 & x_1\\
\cdots & \cdots & \cdots & \cdots & x_2 & 1-x_2 & \cdots & \cdots\\
\cdots & \cdots & \cdots & \cdots & 1-x_2 & x_2 & \cdots & \cdots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
x_m & 1-x_m & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
1-x_m & x_m & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix},
$$

where $\frac{1}{2} \leq x_i \leq 1 - \frac{1}{2m}$.

Then B satisfies the star condition for all $A \in \Omega_n$ such that $T(A[p, p + 1/n - q, n - q + 1] \leq 1$ for all odd $p, 1 \leq p \leq n$ and all odd $q, 1 \leq q \leq n$.

Proof.

$$\sum_{i,j=1}^{n} a_{ij} perB_{ij} - perA - (n-1)perB$$
$$= a_{1,n-1}x_1(2x_2^2 - 2x_2 + 1)\cdots(2x_m^2 - 2x_m + 1) + a_{1,n}(1 - x_1)$$
$$(2x_2^2 - 2x_2 + 1)\cdots(2x_m^2 - 2x_m + 1) + a_{2,n-1}(1 - x_1)$$
$$(2x_2^2 - 2x_2 + 1)\cdots(2x_m^2 - 2x_m + 1) + a_{2,n}x_1(2x_2^2 - 2x_2 + 1)\cdots$$
$$(2x_m^2 - 2x_m + 1)\cdots + a_{n-1,1}x_m(2x_1^2 - 2x_1 + 1)\cdots(2x_m^2 - 2x_m + 1)$$
$$+ (1 - x_m)a_{n-1,2}(2x_1^2 - 2x_1 + 1)\cdots 2x_m^2 - 2x_m + 1) + (1 - x_m)a_{n,1}$$
$$(2x_1^2 - 2x_1 + 1)\cdots(2x_m^2 - 2x_m + 1) + x_m a_{n,2}(2x_1^2 - 2x_1 + 1)\cdots$$
$$(2x_m^2 - 2x_m + 1) - perA - (n-1)(2x_1^2 - 2x_1 + 1)\cdots(2x_m^2 - 2x_m + 1)$$
$$\leq (a_{1,n-1} + a_{1,n} + a_{2,n-1} + a_{2,n})x_1(2x_2^2 - 2x_2 + 1)\cdots(2x_m^2 - 2x_m + 1)$$
$$+ \cdots + (a_{n-1,1} + a_{n-1,2} + a_{n,1} + a_{n,2})x_m(2x_1^2 - 2x_1 + 1)\cdots$$
$$(2x_{m-1}^2 - 2x_m + 1) - perA - (n-1)(2x_1^2 - 2x_1 + 1)\cdots(2x_m^2 - 2x_m + 1)$$
$$\leq x_1(2x_2^2 - 2x_2 + 1)\cdots(2x_m^2 - 2x_m + 1) + x_2(2x_1^2 - 2x_1 + 1)\cdots$$
$$(2x_m^2 - 2x_m + 1)\cdots + x_m(2x_1^2 - 2x_1 + 1)\cdots(2x_{m-1}^2 - 2x_{m-1} + 1)$$
$$- perA - (n-1)(2x_1^2 - 2x_1 + 1)\cdots(2x_m^2 - 2x_m + 1)$$
$$= (2x_2^2 - 2x_2 + 1)\cdots(2x_m^2 - 2x_m + 1)[x_1 - \frac{n-1}{m}(2x_1^2 - 2x_1 + 1)]$$
$$+(2x_1^2 - 2x_1 + 1)\cdots(2x_m^2 - 2x_m + 1)[x_2 - \frac{n-1}{m}(2x_2^2 - 2x_2 + 1)] + \cdots +$$
$$(2x_1^2 - 2x_1 + 1)\cdots(2x_{m-1}^2 - 2x_{m-1} + 1)[x_m - \frac{n-1}{m}(2x_m^2 - 2x_m + 1)] - perA.$$

Now,

$$x_i - \frac{n-1}{m}(2x_i^2 - 2x_i + 1)$$
$$= x_i - \frac{2m-1}{m}(2x_i^2 - 2x_i + 1)$$

$$= x_i + (-2 + \frac{1}{m})(2x_i^2 - 2x_i + 1) \leq x_i + (-2 + \frac{1}{m})\frac{1}{2}$$
$$= x_i - 1 + \frac{1}{2m} \leq 0$$

since $\frac{1}{2} \leq x_i \leq 1 - \frac{1}{2m}$.

Therefore, the inequality (2) is satisfied. □

Theroem 2.3. *Let*

$$B = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix} \bigoplus \begin{pmatrix} y & 1-y \\ 1-y & y \end{pmatrix}$$

where $0 \leq x, y \leq \frac{1}{2}$*. Then* B *satisfies (1) for matrices* $A \in \Omega_4$ *where*

$$A = \begin{pmatrix} a & 0 & 1-a & 0 \\ 0 & a & 0 & 1-a \\ 1-a & 0 & a & 0 \\ 0 & 1-a & 0 & a \end{pmatrix}$$

where $0 \leq a \leq 1$.

Proof. A necessary and sufficient condition for a matrix $B \in \Omega_4$ to be a star is that

$$\sum_{i,j=1}^{4} b_{ij} perA_{ij} - perB - 3perA \leq 0.$$

Now, consider

$$\sum_{i,j=1}^{4} b_{ij} perA_{ij} - perB - 3perA$$
$$= x(a^3 + a(1-a)^2) + x(a^3 + (1-a)^2) + y(a^3 + (1-a)^2)$$
$$+ y(a^3 + (1-a)^2) - perB - 3[a^4 + a^2(1 - 2a + a^2) + (1-a)(a^2(1-a)$$
$$+ (1-a)^3)]$$
$$\leq 2x(a^3 + a(1-a)^2) + 2y(a^3 + a(1-a)^2) - \frac{3}{32} - 3[a^4 + a^2 - 2a^3 + a^4$$

$$
\begin{aligned}
&+ (1-a)(a^2 - a^3 + 1 - 3a + 3a^2 - a^3)] \\
&\le 2(a^3 + a(1-2a+a^2)) - \frac{3}{32} - 3[2a^4 - 2a^3 + a^2 + a^4 - 2a^3 + a^2 + 1 \\
&+ 4a^2 + a^4 - 4a - 4a^3 + 2a^2] \\
&= 4a^3 - 4a^2 + 2a - \frac{3}{32} - 3(4a^4 - 8a^3 + 8a^2 - 4a + 1) \\
&= 4a^3 - 4a^2 + 2a - \frac{3}{32} - 12a^4 + 24a^3 - 24a^2 + 12a - 3 \\
&= -12a^4 + 28a^3 - 28a^2 + 14a - \frac{99}{32} \\
&\le 0. \qquad \square
\end{aligned}
$$

Theorem 2.4. *Let*

$$B = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix} \bigoplus \begin{pmatrix} y & 1-y \\ 1-y & y \end{pmatrix},$$

where $0 \le x, y \le \sqrt{0.375}$*. Then* B *satisfies the condition (2) for all* $A \in \Omega_4$ *of the form*

$$A = \begin{pmatrix} a & 0 & 1-a & 0 \\ 0 & a & 0 & 1-a \\ 1-a & 0 & a & 0 \\ 0 & 1-a & 0 & a \end{pmatrix},$$

where $0 \le a \le \sqrt{0.375}$.

Proof.

$$
\begin{aligned}
&\sum_{i,j=1}^{4} a_{ij} perB_{ij} - perA - 3perB \\
&= 2ax(2y^2 - 2y + 1) + 2ay(2x^2 - 2x + 1) - perA - 3(2y^2 - 2y + 1) \\
&(2x^2 - 2x + 1) \\
&= (2y^2 - 2y + 1)(2ax - \frac{3}{2}(2x^2 - 2x + 1)) + (2x^2 - 2x + 1)\Big(2ay - \frac{3}{2} \\
&(2y^2 - 2y + 1)\Big) - perA.
\end{aligned}
$$

It is easy to see that

$$2ax - \frac{3}{2}(2x^2 - 2x + 1) \le 0$$

and

$$0 \leq 2ax \leq 0.75,$$

since $\frac{1}{2} \leq 2x^2 - 2x + 1 \leq 1$ and $0 \leq x, y, a \leq \sqrt{0.375}$.

Hence the inequality (2) is satisfied. □

Theorem 2.5. *Let*

$$B = \begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix} \bigoplus \begin{pmatrix} y & 1-y \\ 1-y & y \end{pmatrix}.$$

Then B satisfies the star condition for all $A \in \Omega_4$ of the form

$$A = \begin{pmatrix} a & a & 1-2a & 0 \\ a & a & 1-2a & 0 \\ 1-2a & 1-2a & 4a-1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $0 \leq a \leq 0.1875$.

Proof.

$$\sum_{i,j=1}^{4} a_{ij} perB_{ij} - perA - 3perB$$

$$= ax(y^2 + (1-y)^2) + a(1-x)(y^2 + (1-y)^2) + a(1-x)(y^2 + (1-y)^2)$$

$$+ ax(y^2 + (1-y)^2) + (4a-1)y(x^2 + (1-x)^2) + y(x^2 + (1-x)^2)$$

$$- perB - 3(2x^2 - 2x + 1)(2y^2 - 2y + 1)$$

$$= 2ax(y^2 + (1-y)^2) + 2a(1-x)(y^2 + (1-y)^2) + 4ay(x^2 + (1-x)^2)$$

$$- perB - 3(2x^2 - 2x + 1)(2y^2 - 2y + 1)$$

$$= 2a(y^2 + (1-y)^2) + 4ay(x^2 + (1-x)^2) - perB - 3(2x^2 - 2x + 1)$$

$$(2y^2 - 2y + 1)$$

$$= (2a - \frac{3}{2}\left(2x^2 - 2x + 1\right))(2y^2 - 2y + 1) + \left(4ay - \frac{3}{2}\left(2y^2 - 2y + 1\right)\right)$$

$$(2x^2 - 2x + 1) - perB.$$

$$\frac{1}{2} \leq 2x^2 - 2x + 1 \leq 1.$$

Multiplying by $\frac{-3}{2}$ throughout the above inequality,

$$\frac{-3}{4} \geq \frac{-3}{2}(2x^2 - 2x + 1) \geq \frac{-3}{2}.$$

$$2a - \frac{3}{2}(2x^2 - 2x + 1) \leq 2a - \frac{3}{4} \leq 0 \text{ if } 2a \leq \frac{3}{4}$$

i.e $a \leq \frac{3}{8}$.

$$4ay - \frac{3}{2}(2y^2 - 2y + 1) \leq 4ay - \frac{3}{4}.$$

Since $y \leq 1$, $4ay - \frac{3}{4} \leq 4a - \frac{3}{4} \leq 0$ if $4a \leq \frac{3}{4}$ i.e $a \leq \frac{3}{16}$.
Hence the inequality (2) is satisfied. □

3. Conclusion.

Conjectures on star matrices are well known conjectures in the theory of permanents. In this paper, we provided some classes of matrices in Ω_n which satisfy the inequality of the star conjecture. We also provided some classes of matrices in Ω_4 which satisfy the star condition for the direct sum of two 2×2 star matrices.

Acknowledgement. We would like to express our sincere gratitude to the referees for their valuable suggestions and comments which improved the paper.

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(*Received: August, 2021; Revised: October, 2021*)