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The result of arithmetic operations applied on general quadratic fuzzy sets

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Abstract. A general quadratic fuzzy set is a quadratic fuzzy set that may not have maximum value 1. We calculated the Zadeh's max-min composition operator for two general quadratic fuzzy sets. By using parametric operations between two α-cuts which are regions, we generalized the general quadratic fuzzy sets from $\mathbb{R}$ to $\mathbb{R}^2$ and calculated the parametric operations for two generalized 2-dimensional quadratic fuzzy sets.

We show that the parametric operations for two generalized quadratic fuzzy sets defined on $\mathbb{R}^2$ is a generalization of a Zadeh's max-min composition operations for two general quadratic fuzzy sets defined on $\mathbb{R}$.

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1. Introduction

The membership function of quadratic fuzzy number consists of a quadratic function with the maximum value 1. A general quadratic fuzzy set is a quadratic fuzzy set that may not have maximum value 1. In [4], we calculated the extended operations for generalized quadratic fuzzy sets.

In [2], we generalized the quadratic fuzzy numbers from $\mathbb{R}$ to $\mathbb{R}^2$. By defining parametric operations between two α-cuts which are regions,

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we got the parametric operations for two quadratic fuzzy numbers defined on $\mathbb{R}^2$. The results for the parametric operations are the generalization of Zadeh's extended algebraic operations. We proved that the results for the parametric operations became the generalization of Zadeh's extended algebraic operations.

Moreover, in [5], we generalized the general quadratic fuzzy sets from $\mathbb{R}$ to $\mathbb{R}^2$ and calculated the parametric operations for two generalized 2-dimensional quadratic fuzzy sets.

In this paper, we show that the parametric operations for two generalized quadratic fuzzy sets defined on $\mathbb{R}^2$ is a generalization of a Zadeh's max-min composition operations for two general quadratic fuzzy sets defined on $\mathbb{R}$.

2. Zadeh's max-min composition operations for generalized quadratic fuzzy sets defined on $\mathbb{R}$

We begin by defining α-cut and α-set of the fuzzy set A on $\mathbb{R}$ with the membership function $\mu_A(x)$.

An *α-cut* of the fuzzy number A is defined by

$$A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\} \text{ if } \alpha \in (0,1]$$

and

$$A_0 = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > 0\}.$$

For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > 0\}$ and $A^1 = A_1$.

Definition 2.1 [8]. The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$ and extended division $A(/)B$ are fuzzy sets with membership functions as follows. For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\}, \quad * = +, -, \cdot, /$$

We now generalize the quadratic fuzzy set. A general quadratic fuzzy set is symmetric and may not have maximum value 1. The graph of membership function of symmetric general quadratic fuzzy set is symmetric with respect to some line $x = m$.

Definition 2.2 [4]. A fuzzy set A with a membership function

$$\mu_A(x) = \begin{cases} 0, & x < x_1,\ x_2 \le x, \\ -a(x-x_1)(x-x_2) = -a(x-m)^2 + p, & x_1 \le x < x_2, \end{cases}$$

where $m = \frac{x_1+x_2}{2}$, $0 < a$, $0 < p \le 1$, is called a *generalized quadratic fuzzy set* and denoted by $[[x_1, p, x_2]]$ or $[[a, m, p]]_+$.

Theorem 2.3 [4]. *Let*

$$A = [[x_1, p, x_2]] = [[a, m, p]]_+$$

and

$$B = [[x_3, q, x_4]] = [[b, n, q]]_+$$

be generalized quadratic fuzzy sets.

Suppose $p \le q$ *and* $\mu_B(x) \ge p$ *on* $[k_1, k_2]$. *Then we have the followings:*

(1) $A(+)B$ *is a fuzzy set with a membership function*

$$\mu_{A(+)B}(x) = \begin{cases} 0 & (x < x_1 + x_3,\ x_2 + x_4 \le x) \\ f_1(x) & (x_1 + x_3 \le x < m + k_1) \\ p & (m + k_1 \le x < m + k_2) \\ f_2(x) & (m + k_2 \le x < x_2 + x_4) \end{cases}$$

where

$$f_1(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(a + b + an + bn) - abn(am + bm + an + bn) - ab(p + q) + a^2q + b^2p + 2ab(am + bm + an + bn)x - ab(a + b)x^2 + 2ab(m + n - x) \cdot \sqrt{g_1(x)}\Big),$$

$$f_2(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(a + b + an + bn) - abn(am + bm + an + bn) - ab(p + q) + a^2q + b^2p + 2ab(am + bm + an + bn)x - ab(a + b)x^2 - 2ab(m + n - x) \cdot \sqrt{g_1(x)}\Big),$$

and

$$g_1(x) = ab(m + n)^2 + (a - b)(p - q) - 2ab(m + n)x + abx^2.$$

(2) *$A(-)B$ is a fuzzy set with a membership function*

$$\mu_{A(-)B}(x) = \begin{cases} 0 & (x < x_1 - x_4, x_2 - x_3 \le x) \\ f_3(x) & (x_1 - x_4 \le x < m - k_2) \\ p & (m - k_2 \le x < m - k_1) \\ f_4(x) & (m - k_1 \le x < x_2 - x_3) \end{cases}$$

where

$$f_3(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(am + bm - an - bn) - abn(an + bn - am - bm) - ab(p + q) + a^2q + b^2p + 2ab(am + bm - an - bn)x - ab^2x^2 + 2ab(m - n - x) \cdot \sqrt{g_2(x)}\Big),$$

$$f_4(x) = \frac{1}{a^2 - 2ab + b^2}\Big(-abm(am + bm - an - bn) - abn(an + bn - am - bm) - ab(p + q) + a^2q + b^2p + 2ab(am + bm - an - bn)x - ab^2x^2 - 2ab(m - n - x) \cdot \sqrt{g_2(x)}\Big),$$

and

$$g_2(x) = ab(m - n)^2 + (a - b)(p - q) - 2ab(m - n)x + abx^2.$$

(3) *If $p = q$, $A(\cdot)B$ is a fuzzy set with a membership function*

$$\mu_{A(\cdot)B}(x) = \begin{cases} 0 & (x < x_1x_3, \ \ x_2x_4 \le x) \\ f_5(x) & (x_1x_3 \le x < x_2x_4) \end{cases}$$

where

$$f_5(x) = \frac{1}{2}(-am^2 - bn^2 + 2p) - \sqrt{ab}x + \frac{1}{2}\sqrt{g_3(x)},$$

$$\begin{aligned} g_3(x) =& - am^2(am^2 + 3bn^2) - bn^2(bn^2 + 3am^2) + 2(am^2 + bn^2 \\ & - 2p)^2 + 8p(am^2 + bn^2 - p) + 8abmnx - \frac{1}{8\sqrt{ab}x}\Big\{-8(am^2 \\ & + bn^2 - 2p)^3 + 8(am^2 + bn^2 - 2p)h_1(x) - 16h_2(x)\Big\}, \\ h_1(x) =& am^2(am^2 + 2bn^2) + bn^2(bn^2 + 2am^2) - 6p(am^2 + bn^2 - p) \\ & - 4abmnx - 2abx^2, \\ h_2(x) =& abm^2n^2(am^2 + bn^2 - 4p) - am^2p(am^2 - 3p) \\ & - bn^2p(bn^2 - 3p) - 2p^3 - 2abmn(am^2 + bn^2 - 2p)x \\ & + ab(am^2 + bn^2 + 2p)x^2. \end{aligned}$$

(4) *$A(/)B$ is a fuzzy set with a membership function*

$$\mu_{A(/)B}(x) = \begin{cases} 0 & (x < x_1/x_4, \ \ x_2/x_3 \le x) \\ f_6(x) & (x_1/x_4 \le x < m/k_2) \\ p & (m/k_2 \le x < m/k_1) \\ f_7(x) & (m/k_1 \le x < x_2/x_3) \end{cases}$$

where

$$\begin{aligned} f_6(x) =& \frac{1}{b^2 - 2abx^2 + a^2x^4}\Big(-b^2(am^2 + p) + 2ab^2mnx - ab(am^2 \\ & + bn^2 + p + q)x^2 + 2a^2bmnx^3 - a^2(bn^2 - q)x^4 \\ & + 2abx(m - nx) \cdot \sqrt{g_4(x)}\Big), \\ f_7(x) =& \frac{1}{b^2 - 2abx^2 + a^2x^4}\Big(-b^2(am^2 + p) + 2ab^2mnx - ab(am^2 \\ & + bn^2 + p + q)x^2 + 2a^2bmnx^3 - a^2(bn^2 - q)x^4 \\ & - 2abx(m - nx) \cdot \sqrt{g_4(x)}\Big), \end{aligned}$$

and

$$g_4(x) = b(am^2 - p + q) - 2abmnx + a(bn^2 + p - q)x^2.$$

Proof. It is enough to calculate the α-cut of $A(+)B$, $A(-)B$, $A(\cdot)B$ and $A(/)B$. Note that

$$\mu_A(x) = \begin{cases} 0 & (x < x_1, \quad x_2 \le x) \\ -a(x - x_1)(x - x_2) & \\ \quad = -a(x - m)^2 + p & (x_1 \le x < x_2) \end{cases}$$

and

$$\mu_B(x) = \begin{cases} 0 & (x < x_3, \quad x_4 \le x) \\ -b(x - x_3)(x - x_4) & \\ \quad = -b(x - n)^2 + q & (x_3 \le x < x_4) \end{cases}$$

Let

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}]$$

and

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}]$$

be the α-cuts of A and B, respectively.

Since

$$\alpha = -a(a_1^{(\alpha)} - m)^2 + p = -a(a_2^{(\alpha)} - m)^2 + p,$$

we have

$$A_\alpha = [a_1^{(\alpha)}, a_2^{(\alpha)}] = \left[m - \sqrt{\frac{p - \alpha}{a}}\ , \ \ m + \sqrt{\frac{p - \alpha}{a}}\ \right].$$

Similarly, we have

$$B_\alpha = [b_1^{(\alpha)}, b_2^{(\alpha)}] = \left[n - \sqrt{\frac{q - \alpha}{b}}\ , \ \ n + \sqrt{\frac{q - \alpha}{b}}\ \right].$$

(1) Since $p \le q$ and $\mu_B(x) \ge p$ on $[k_1, k_2]$, $\mu_{A(+)B}(x) = p$ if $x \in [m +$

$k_1, m + k_2]$. By the above facts,

$$\begin{aligned} A_\alpha(+)B_\alpha &= [a_1^{(\alpha)} + b_1^{(\alpha)}, a_2^{(\alpha)} + b_2^{(\alpha)}] \\ &= \Big[m - \sqrt{\frac{p-\alpha}{a}} + n - \sqrt{\frac{q-\alpha}{b}}, \\ &\qquad m + \sqrt{\frac{p-\alpha}{a}} + n + \sqrt{\frac{q-\alpha}{b}}\ \Big]. \end{aligned}$$

If $x \in [x_1 + x_3, m + k_1]$, then $m - \sqrt{\frac{p-\alpha}{a}} + n - \sqrt{\frac{q-\alpha}{b}} = x$.

Hence $\alpha = f_1(x)$. For $x \in [m + k_2, x_2 + x_4]$, we have $\alpha = f_2(x)$.

In cases of (2), (3), and (4), we can prove similarly. □

3. Parametric operations for generalized 2-dimensional quadratic fuzzy sets on $\mathbb{R}^2$

In this section, we define the generalized 2-dimensional quadratic fuzzy sets on $\mathbb{R}^2$. We defined the parametric operations between two 2-dimensional quadratic fuzzy sets using the operations between α-sets in $\mathbb{R}^2$. The α-set is region in $\mathbb{R}^2$.

We interpret the existing method from a different perspective and apply the method to α-sets which are regions in $\mathbb{R}^2$.

Definition 3.1. A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} h - \big(\frac{(x-x_1)^2}{a^2} + \frac{(y-y_1)^2}{b^2}\big), & \\ & \text{if } \ b^2(x-x_1)^2 + a^2(y-y_1)^2 \le ha^2b^2, \\ 0, & \text{otherwise,} \end{cases}$$

where $a, b > 0$, $x_1 - a \le x \le x_1 + a$, $y_1 - b \le y \le y_1 + b$, and $0 < h < 1$ is called the *the generalized 2-dimensional quadratic fuzzy set* and denoted by $[[a,\ x_1,\ h,\ b,\ y_1]]^2$.

Note that $\mu_A(x,y)$ is a cone. The intersections of $\mu_A(x,y)$ and the horizontal planes $z = \alpha$ $(0 < \alpha < 1)$ are ellipses. The intersections of $\mu_A(x,y)$

and the vertical planes $y-y_1 = k(x-x_1)$ $(k \in \mathbb{R})$ are symmetric quadratic fuzzy sets in those planes. If $a = b$, ellipses become circles.

The α-cut A_α of a generalized 2-dimensional quadratic fuzzy set $A = [[a, x_1, h, b, y_1]]^2$ is the interior of the ellipse in xy-plane including the boundary

$$A_\alpha = \Big\{(x, y) \in \mathbb{R}^2 \;\Big|\; \frac{(x-x_1)^2}{a^2(h-\alpha)} + \frac{(y-y_1)^2}{b^2(h-\alpha)} \leq 1\Big\}.$$

Theorem 3.2 [3]. *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^2$ and $A^\alpha = \{(x, y) \in \mathbb{R}^2 | \mu_A(x, y) = \alpha\}$ be the α-set of A. Then for all $\alpha \in (0, 1)$, there exist continuous functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$ defined on $[0, 2\pi]$ such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \leq t \leq 2\pi\}.$$

Proof. Let $\alpha \in (0, 1)$ be fixed. Since A is a convex fuzzy number defined on $\mathbb{R}^2$, the α-cut A_α is convex subset in $\mathbb{R}^2$. Let

$$l = \inf\{x | \mu_A(x, y) = \alpha\} \text{ and } m = \sup\{x | \mu_A(x, y) = \alpha\}.$$

The upper boundary of A_α is the graph of a piecewise continuous concave function $h_1(x)$ and the lower boundary of A_α is also the graph of a piecewise continuous convex function $h_2(x)$ defined on $[l, m]$.

Since $h_1(x)$ is piecewise continuous, $h_1(x)$ is continuous on $[l, m]$ except finitely many points $l < x_n < x_{n-1} < \cdots < x_1 < m$.

Note that x_1 and x_n may equal to the end points m and l, respectively.

Similarly, since $h_2(x)$ is also piecewise continuous, $h_2(x)$ is continuous on $[l, m]$ except finitely many points $l < x_{n+1} < x_{n+2} < \cdots < x_{n+m} < m$.

Note that x_{n+1} and x_{n+m} may equal to the end points l and m, respectively. If the end points l and m(or one of them) equal to some x_i, we can prove the above facts similarly. Define

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \quad \text{if} \quad t \in [0, \pi],$$

except the points

$$t_i = \cos^{-1}\left(\frac{2(x_i - m)}{m-l} + 1\right), \ \ i = 1, 2, \cdots, n.$$

Then $f_1^\alpha(t)$ is piecewise continuous on $[0, \pi]$ and

$$\{l \le x \le m | x \ne x_i, i = 1, 2, \cdots, n\}$$
$$= \{f_1^\alpha(t) | t \in [0, \pi], t \ne t_i, i = 1, 2, \cdots, n\}.$$

Define

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \quad \text{if} \quad t \in [\pi, 2\pi],$$

except the points

$$t_j = \cos^{-1}\left(\frac{2(x_{n+j} - m)}{m-l} + 1\right), \ \ j = 1, 2, \cdots, m.$$

Then $f_1^\alpha(t)$ is piecewise continuous on $[\pi, 2\pi]$ and

$$\{l \le x \le m | x \ne x_{n+j}, j = 1, 2, \cdots, m\}$$
$$= \{f_1^\alpha(t) | t \in [\pi, 2\pi], t \ne t_{n+j}, j = 1, 2, \cdots, m\}.$$

The explicit proof for piecewise continuous can be proved by the same way in the proof of Theorem 3.2([1]). Focussing the construction of functions $f_1^\alpha(t)$ and $f_2^\alpha(t)$, we outline our proof.

Define $f_1^\alpha(t)$ and $f_2^\alpha(t)$ by

$$f_1^\alpha(t) = \frac{1}{2}(m-l)(\cos t - 1) + m, \quad t \in [0, 2\pi],$$

and

$$f_2^\alpha(t) = \begin{cases} h_1(f_1^\alpha(t)), & 0 \le t \le \pi, \\ h_2(f_1^\alpha(t)), & \pi \le t \le 2\pi. \end{cases}$$

Then we have

$$A^{\alpha} = \{(f_1^{\alpha}(t), f_2^{\alpha}(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

The proof is complete. □

Definition 3.3. Let A and B be convex fuzzy sets defined on $\mathbb{R}^2$ and

$$A^{\alpha} = \{(x,y) \in \mathbb{R}^2 | \mu_A(x,y) = \alpha\} = \{(f_1^{\alpha}(t), f_2^{\alpha}(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$
$$B^{\alpha} = \{(x,y) \in \mathbb{R}^2 | \mu_B(x,y) = \alpha\} = \{(g_1^{\alpha}(t), g_2^{\alpha}(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

be the α-sets of A and B, respectively.

For $\alpha \in (0,1)$, we define that the parametric addition $A(+)_pB$, parametric subtraction $A(-)_pB$, parametric multiplication $A(\cdot)_pB$ and parametric division $A(/)_pB$ of two fuzzy sets A and B are fuzzy sets that have their α-sets as follows:

(1) $A(+)_pB$:

$$(A(+)_pB)^{\alpha} = \{(f_1^{\alpha}(t) + g_1^{\alpha}(t), f_2^{\alpha}(t) + g_2^{\alpha}(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(2) $A(-)_pB$:

$$(A(-)_pB)^{\alpha} = \{(x_{\alpha}(t), y_{\alpha}(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_{\alpha}(t) = \begin{cases} f_1^{\alpha}(t) - g_1^{\alpha}(t+\pi), & \text{if } 0 \le t \le \pi \\ f_1^{\alpha}(t) - g_1^{\alpha}(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

and

$$y_{\alpha}(t) = \begin{cases} f_2^{\alpha}(t) - g_2^{\alpha}(t+\pi), & \text{if } 0 \le t \le \pi \\ f_2^{\alpha}(t) - g_2^{\alpha}(t-\pi), & \text{if } \pi \le t \le 2\pi \end{cases}$$

(3) $A(\cdot)_pB$:

$$(A(\cdot)_pB)^{\alpha} = \{(f_1^{\alpha}(t) \cdot g_1^{\alpha}(t), f_2^{\alpha}(t) \cdot g_2^{\alpha}(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}$$

(4) $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$

where

$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t+\pi)} \quad (0 \le t \le \pi),$$
$$x_\alpha(t) = \frac{f_1^\alpha(t)}{g_1^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

and

$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t+\pi)} \quad (0 \le t \le \pi),$$
$$y_\alpha(t) = \frac{f_2^\alpha(t)}{g_2^\alpha(t-\pi)} \quad (\pi \le t \le 2\pi)$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha \to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha \to 1^-}(A(*)_pB)^\alpha$, where $* = +, \ -, \ \cdot, \ /$.

Theorem 3.4 [5]. *Let*

$$A = [[a_1, \ x_1, \ h_1, \ b_1, \ y_1]]^2$$

and

$$B = [[a_2, \ x_2, h_2, \ b_2, \ y_2]]^2 \quad (0 < h_1 < h_2 < 1)$$

be two generalized 2-dimensional quadratic fuzzy sets. Since A and B are convex fuzzy sets defined on $\mathbb{R}^2$*, by Theorem 3.2, there exists* $f_i^\alpha(t), g_i^\alpha(t)$ $(i = 1, 2)$ *such that*

$$A^\alpha = \{(f_1^\alpha(t), f_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\},$$
$$B^\alpha = \{(g_1^\alpha(t), g_2^\alpha(t)) \in \mathbb{R}^2 | 0 \le t \le 2\pi\}.$$

Since

$$A = [[a_1, \ x_1, \ h_1, \ b_1, \ y_1]]^2$$

and

$$B = [[a_2,\ x_2,\ h_2,\ b_2,\ y_2]]^2,$$

we have

$$\begin{aligned} f_1^\alpha(t) &= x_1 + a_1\sqrt{h_1-\alpha}\cos t,\\ f_2^\alpha(t) &= y_1 + b_1\sqrt{h_1-\alpha}\sin t,\\ g_1^\alpha(t) &= x_2 + a_2\sqrt{h_2-\alpha}\cos t,\\ g_2^\alpha(t) &= y_2 + b_2\sqrt{h_2-\alpha}\sin t. \end{aligned}$$

For $0 < \alpha < h_1$, *we have the following:*

(1) $(A(+)_pB)^\alpha$:

$$\begin{aligned} (A(+)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2 \Big| &\Big(\frac{x-x_1-x_2}{a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha}}\Big)^2 \\ &+\Big(\frac{y-y_1-y_2}{b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha}}\Big)^2 = 1\Big\}. \end{aligned}$$

(2) $(A(-)_pB)^\alpha$:

$$\begin{aligned} (A(-)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2 \Big| &\Big(\frac{x-x_1+x_2}{a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha}}\Big)^2 \\ &+\Big(\frac{y-y_1+y_2}{b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha}}\Big)^2 = 1\Big\}. \end{aligned}$$

(3) $(A(\cdot)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$:

$$\begin{aligned} x_\alpha(t) &= x_1x_2 + (x_1a_2\sqrt{h_2-\alpha} + x_2a_1\sqrt{h_1-\alpha})\cos t\\ &\quad + a_1a_2\sqrt{h_1-\alpha}\sqrt{h_2-\alpha}\cos^2 t,\\ y_\alpha(t) &= y_1y_2 + (y_1b_2\sqrt{h_2-\alpha} + y_2b_1\sqrt{h_1-\alpha})\sin t\\ &\quad + b_1b_2\sqrt{h_1-\alpha}\sqrt{h_2-\alpha}\sin^2 t. \end{aligned}$$

(4) $(A(/)_pB)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$:

$$x_\alpha(t) = \frac{x_1 + a_1\sqrt{h_1-\alpha}\cos t}{x_2 - a_2\sqrt{h_2-\alpha}\cos t}$$

and

$$y_\alpha(t) = \frac{y_1 + b_1\sqrt{h_1-\alpha}\sin t}{y_2 - b_2\sqrt{h_2-\alpha}\sin t}.$$

If $\alpha = h_1$, *we have*

$$(A(*)_pB)^{h_1} = \lim_{\alpha\to h_1^-}(A(*)_pB)^\alpha, \quad * = +,\ -,\ \cdot,\ /,$$

and for $h_1 < \alpha \leq h_2$, *by the Zadeh's max-min principle operations, we have to define*

$$(A(*)_pB)^\alpha = \emptyset, \quad * = +,\ -,\ \cdot,\ /.$$

Proof. (1) Let $0 < \alpha < h_1$. Since

$$f_1^\alpha(t) + g_1^\alpha(t) = x_1 + x_2 + (a_1\sqrt{h_1-\alpha} + a_2\sqrt{h_2-\alpha})\cos t$$

and

$$f_2^\alpha(t) + g_2^\alpha(t) = y_1 + y_2 + (b_1\sqrt{h_1-\alpha} + b_2\sqrt{h_2-\alpha})\sin t,$$

we have

$$(A(+)_pB)^\alpha = \Big\{(x,y)\in\mathbb{R}^2 \Big| \Big(\frac{x-x_1-x_2}{a_1\sqrt{h_1-\alpha}+a_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{y-y_1-y_2}{b_1\sqrt{h_1-\alpha}+b_2\sqrt{h_2-\alpha}}\Big)^2 = 1\Big\}.$$

In cases of (2), (3), and (4), we can prove similarly. □

4. A generalization of a general quadratic fuzzy sets

In this section, we show that the parametric operations for two generalized quadratic fuzzy sets defined on $\mathbb{R}^2$ is a generalization of a Zadeh's max-min composition operations for two general quadratic fuzzy sets defined on $\mathbb{R}$. For that, we have to prove that the intersections of the results on $\mathbb{R}^2$ and vertical plane are as same as those on $\mathbb{R}$.

Theorem 4.1. *For* $* = +, -, \cdot, /$, *let* $\mu_{A(*)B}(x,y)$ *be the results in Theorem 3.4 and* $\mu_{A(*)B}(x)$ *be the results in Theorem 2.3.*

Let $\mu^1_{A(*)B}(x,0)$ *be the fuzzy sets on xz-plane such that*

$$\mu_{A(*)B}(x,0) \cap \{xz - \text{plane}\} = \mu^1_{A(*)B}(x,0).$$

Then we have

$$\mu^1_{A(*)B}(x,0) = \mu_{A(*)B}(x).$$

Proof. In Theorem 2.3, A and B are symmetric generalized quadratic fuzzy sets. Thus we consider only the symmetric case in Theorem 3.4. Let $a_1 = b_1$ and $a_2 = b_2$ in Theorem 3.4.

Let

$$A = [[a_1,\ x_1,\ h_1,\ a_1,\ y_1]]^2 \equiv [[a_1,\ x_1,\ h_1,\ y_1]]^2$$

and

$$B = [[a_2,\ x_2,\ h_2,\ a_2, y_2]]^2 \equiv [[a_2,\ x_2,\ h_2,\ y_2]]^2.$$

Since $y_1 = y_2 = 0$, $A = [[a_1,\ x_1,\ h_1,\ 0]]^2$ and $B = [[a_2,\ x_2,\ h_2,\ 0]]^2$.

Thus we have

$$\mu_A(x,y) = \begin{cases} h_1 - \left(\frac{(x-x_1)^2}{a_1^2} + \frac{y^2}{a_1^2}\right), & (x-x_1)^2 + y^2 \le h_1 a_1^2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} A_\alpha &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; (x-x_1)^2 + y^2 \le a_1^2(h_1-\alpha)\right\} \\ &= \left\{(x,y) \in \mathbb{R}^2 \;\middle|\; \left(\frac{x-x_1}{a_1\sqrt{h_1-\alpha}}\right)^2 + \left(\frac{y}{a_1\sqrt{h_1-\alpha}}\right)^2 \le 1\right\}. \end{aligned}$$

Similarly,

$$\mu_B(x,y) = \begin{cases} h_2 - \left(\frac{(x-x_2)^2}{a_2^2} + \frac{y^2}{a_2^2}\right), & (x-x_2)^2 + y^2 \le h_2 a_2^2, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$B_\alpha = \Big\{(x,y)\in\mathbb{R}^2 \;\Big|\; (x-x_2)^2+y^2 \le a_2^2(h_2-\alpha)\Big\}$$
$$= \Big\{(x,y)\in\mathbb{R}^2 \;\Big|\; \Big(\frac{x-x_2}{a_2\sqrt{h_2-\alpha}}\Big)^2 + \Big(\frac{y}{a_2\sqrt{h_2-\alpha}}\Big)^2 \le 1\Big\}.$$

We calculate the end points of 0-cuts and h_1-cuts of $\mu^1_{A(*)B}(x,0)$ and $\mu_{A(*)B}(x)$.

(1) If $y=0$, by Theorem 3.4,

$$\lim_{\alpha\to 0^+}(A(+)_pB)^\alpha = \Big\{(x,0)\in\mathbb{R}^2 \;\Big|\; \Big(\frac{x-x_1-x_2}{a_1\sqrt{h_1}+a_2\sqrt{h_2}}\Big)^2 = 1\Big\}.$$

Thus

$$x = x_1+x_2 \pm (a_1\sqrt{h_1}+a_2\sqrt{h_2}). \tag{4.1}$$

If

$$\mu_A(x,y)=\mu_B(x,y)=0$$
$$,(x-x_1)^2+y^2 = h_1a_1^2 \quad \text{and}$$
$$(x-x_2)^2+y^2 = h_2a_2^2.$$

Thus if $y=0$, we have

$$x = x_1 \pm a_1\sqrt{h_1}, \qquad x = x_2 \pm a_2\sqrt{h_2}. \tag{4.2}$$

From (4.1) and (4,2), the 0-cut of $\mu^1_{A(+)B}(x,0)$ is equal to the 0-cut of $\mu_{A(+)B}(x)$ in Theorem 2.3. If $y=0$, by Theorem 3.4,

$$\lim_{\alpha\to h_1^-}(A(+)_pB)^\alpha = \Big\{(x,0)\in\mathbb{R}^2 \;\Big|\; \Big(\frac{x-x_1-x_2}{a_2\sqrt{h_2-h_1}}\Big)^2 = 1\Big\}.$$

Thus

$$x = x_1+x_2 \pm (a_2\sqrt{h_2-h_1}). \tag{4.3}$$

To avoid confusion, for x_i $(i=1,2,3,4)$ in Theorem 2.3, put $z_i = x_i$ $(i=1,2,3,4)$. To sum up, we have

$$\begin{cases} a=\frac{1}{a_1^2}, z_1 = x_1-a_1\sqrt{h_1}, z_2 = x_1+a_1\sqrt{h_1}, m = x_1, \quad p = h_1, \\ b=\frac{1}{a_2^2}, z_3 = x_2-a_2\sqrt{h_2}, z_4 = x_2+a_2\sqrt{h_2}, n = x_2, \quad q = h_2. \end{cases} \tag{4.4}$$

If $y = 0$ and $\alpha = h_1$, $B^{h_1} = \{(x, y) \in \mathbb{R}^2 \mid (x - x_2)^2 = a_2^2(h_2 - h_1)\}$. Thus $x = x_2 \pm a_2\sqrt{h_2 - h_1}$. This means that k_1 and k_2 in Theorem 2.3 are $k_1 = x_2 - a_2\sqrt{h_2 - h_1}$ and $k_2 = x_2 + a_2\sqrt{h_2 - h_1}$. Therefore

$$\begin{cases} m + k_1 = x_1 + x_2 - a_2\sqrt{h_2 - h_1} \\ m + k_2 = x_1 + x_2 + a_2\sqrt{h_2 - h_1} \end{cases} \tag{4.5}$$

From (4.3) and (4,5), the h_1-cut of $\mu^1_{A(+)B}(x, 0)$ is equal to the h_1-cut of $\mu_{A(+)B}(x)$ in Theorem 2.3.

Putting $y = 0$ in $(A(+)_pB)^\alpha$, we have

$$\Big\{(x, 0) \in \mathbb{R}^2 \ \Big|\ \Big(\frac{x - x_1 - x_2}{a_1\sqrt{h_1 - \alpha} + a_2\sqrt{h_2 - \alpha}}\Big)^2 = 1\Big\} \tag{4.6}$$

Let $f(x)$ be a function such that $\alpha = f(x)$ satisfies equation (4.6). For function $f_1(x)$ in Theorem 2.3, the proof is complete if we prove $f(x) = f_1(x)$. Since there is an error in calculating function $f_1(x), f_1(x)$ needs to be replaced with $F_1(x)$.

$$\begin{aligned} F_1(x) = \frac{1}{a-b}\Big(&\frac{G(x)}{a-b}(m + n - x) - bp + aq - abm^2 - abx^2 \\ &- 2abmn + 2abnx - abn^2 + 2abmx\Big), \end{aligned}$$

where

$$\begin{aligned} G(x) &= 2ab(\sqrt{H(x)} - bm - bn + bx), \\ H(x) &= ap + 2abmn - aq + abm^2 + abx^2 - 2abmx - 2abnx + abn^2 \\ &\quad - bp + bq. \end{aligned}$$

Therefore, we need to prove that $f(x) = F_1(x)$.

It is hard to prove that $f(x)$ and $F_1(x)$ are the same, since they are greatly different in terms of forms. However, all we need to prove is that both functions share the same result in the interval $[x_1 + x_3,\ m + k_1]$.

By using Mathematica, we compared the graphs in the interval $[x_1 + x_3,\ m + k_1]$ and saw that they are identical. For further proof, we also need to revise and compare the functions, but for now, we will prove that 0-cuts and h_1-cuts are the same, respectively.

(2) If $y = 0$, by Theorem 3.4,

$$\lim_{\alpha\to 0^+} (A(-)_p B)^\alpha = \Big\{(x,0) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 + x_2}{a_1\sqrt{h_1} + a_2\sqrt{h_2}}\Big)^2 = 1\Big\}.$$

$$x = x_1 - x_2 \pm (a_1\sqrt{h_1} + a_2\sqrt{h_2}). \tag{4.7}$$

By (4.4), we have

$$\begin{cases} z_1 - z_4 = x_1 - x_2 - (a_1\sqrt{h_1} + a_2\sqrt{h_2}), \\ z_2 - z_3 = x_1 - x_2 + (a_1\sqrt{h_1} + a_2\sqrt{h_2}). \end{cases} \tag{4.8}$$

From (4.7) and (4,8), the 0-cut of $\mu^1_{A(-)B}(x,0)$ is equal to the 0-cut of $\mu_{A(-)B}(x)$ in Theorem 2.3. If $y = 0$, by Theorem 3.4,

$$\lim_{\alpha\to h_1^-} (A(-)_p B)^\alpha = \Big\{(x,0) \in \mathbb{R}^2 \;\Big|\; \Big(\frac{x - x_1 + x_2}{a_2\sqrt{h_2 - h_1}}\Big)^2 = 1\Big\}.$$

Thus

$$x = x_1 - x_2 \pm (a_2\sqrt{h_2 - h_1}). \tag{4.9}$$

By (4.4), we have

$$m - k_2 = x_1 - x_2 - a_2\sqrt{h_2 - h_1}, \quad m - k_1 = x_1 - x_2 + a_2\sqrt{h_2 - h_1}. \tag{4.10}$$

From (4.9) and (4,10), the h_1-cut of $\mu^1_{A(-)B}(x,0)$ is equal to the h_1-cut of $\mu_{A(-)B}(x)$ in Theorem 2.3.

(3) Since $x_\alpha(t) = (x_1 + a_1\sqrt{h_1 - \alpha}\cos t)(x_2 + a_2\sqrt{h_2 - \alpha}\cos t)$, by (4.4),

$$x_0(\pi) = (x_1 - a_1\sqrt{h_1})(x_2 - a_2\sqrt{h_2}) = z_1 z_3.$$

Similarly, by (4.4),

$$\begin{aligned} x_0(0) = x_0(2\pi) &= (x_1 + a_1\sqrt{h_1})(x_2 + a_2\sqrt{h_2}) = z_2 z_4, \\ x_{h_1}(\pi) &= x_1(x_2 - a_2\sqrt{h_2 - h_1}) = mk_1, \\ x_{h_1}(0) = x_{h_1}(2\pi) &= x_1(x_2 + a_2\sqrt{h_2 - h_1}) = mk_2. \end{aligned}$$

Thus the 0-cut is $[x_1x_3,\ x_2x_4]$ and the h_1-cut is $[mk_1,\ mk_2]$. The result (3) of Theorem 2.3 should be corrected.

(4) Since $x_{h_1}(t) = \dfrac{x_1}{x_2 - a_2\sqrt{h_2 - h_1}\cos t}$, by (4.4),

$$\lim_{t\to\pi^-} x_{h_1}(t) = \frac{x_1}{x_2 + a_2\sqrt{h_2 - h_1}} = \frac{m}{k_2},$$
$$\lim_{t\to 0^+} x_{h_1}(t) = \lim_{t\to 2\pi^-} x_{h_1}(t)\frac{x_1}{x_2 - a_2\sqrt{h_2 - h_1}} = \frac{m}{k_1},$$
$$x_0(0) = x_0(2\pi) = \frac{x_1 + a_1\sqrt{h_1}}{x_2 - a_2\sqrt{h_2}} = \frac{z_2}{z_3}.$$

Thus the 0-cut and h_1-cut of $\mu^1_{A(/)B}(x,0)$ are equal to the 0-cut and the h_1-cut of $\mu_{A(/)B}(x)$ in Theorem 2.3, respectively. □

For example, if $A = [[6, 3, \frac{1}{2}, 8, 5]]^2$ and $B = [[4, 6, \frac{2}{3}, 6, 8]]^2$, we have the following graphs $A(+)_pB$ and $A(-)_pB$.

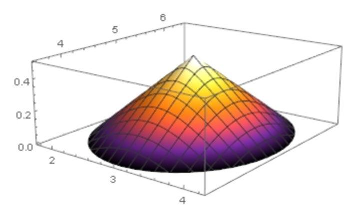

Figure 1: A

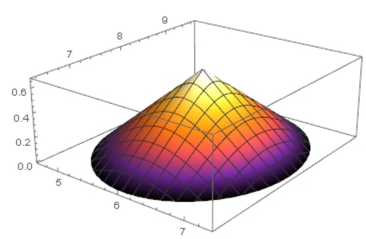

Figure 2: B

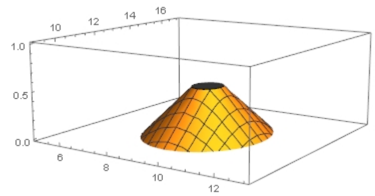

Figure 3: $A(+)_p B$

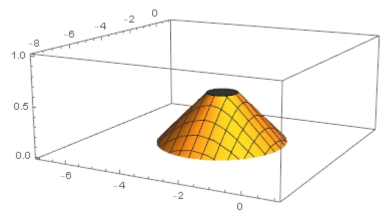

Figure 4: $A(-)_p B$

5. Conclusion

In [4] and [5], we had defined quadratic fuzzy sets and general quadratic fuzzy sets on $\mathbb{R}$ and $\mathbb{R}^2$, and we had obtained the results of arithmetic operations defined using Zadeh's max-min composition operations applied on these fuzzy sets.

In this paper, we proved that results of arithmetic operations applied on general quadratic fuzzy sets on $\mathbb{R}^2$ is a generalization of results of arithmetic operations applied on general quadratic fuzzy sets on $\mathbb{R}$. That is, we proved that the section of results of arithmetic operations applied on a general quadratic fuzzy sets on $\mathbb{R}^2$ is the same as the results of arithmetic operations applied on a section of a general quadratic fuzzy sets on $\mathbb{R}^2$, which is a general quadratic fuzzy set on $\mathbb{R}$.

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